Online Appendix to Risk and Return in Segmented Markets with Expertise

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1 Static Model

We present a static model to build intuition about the interaction between the size and expertise distribution of investors and equilibrium returns.

1.1 Model and Results

Setup Investors have constant relative risk aversion preferences over date 1 consumption, with coefficient of relative risk aversion γ. At date 0, they are endowed with financial wealth W and expertise X. There is a riskless asset with gross return R_f, and a risky asset with gross returns R, which are distributed log normally. We use lower case letters to denote logs.

We assume that the log return on the risky asset for any given investor, which we denote by r, is distributed according to $r \sim N(\boldsymbol{\mu} - \frac{1}{2}\frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X})$, given the distribution of W and X. We denote the variance of log returns on the fundamental asset, before expertise is applied, by σ_v^2, and call this fundamental variance, and its square root fundamental volatility. The effective variance and volatility of an investor's return on the risky asset then decreases as expertise X increases, while the innovation v itself is independent from W and X. We provide an example microfoundation for a closely related return process in the context of our dynamic model in the Appendix.

Investing in the complex asset implies a joint investment in a common market clearing return, as well as a specific risk from hedging or asset specificities. We assume the specified

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functional form for log return volatility for simplicity, as it allows for straightforward calculations of all expectations, and minimal parameters. It is straightforward to show that our main conclusions for the static model are robust to a family of functions $\frac{\sigma_v^2}{k_0+k_1X+k_2X^2+...}$, with all coefficients $k_0, k_1, k_2 ....$ being non-negative. In levels, expected returns $\boldsymbol{\mu}$ are the same for all investors, regardless of their individual expertise.

Solving the Portfolio Choice Problem Using the approximation described in **?**, and the associated appendix **?**, which relates log individual-asset returns to log portfolio returns over short time intervals, the investor's optimization problem becomes:

$$\max_{\theta} \left\{ \theta \left(\boldsymbol{\mu} - r_f \right) - \frac{\gamma}{2} \theta^2 \frac{\sigma_v^2}{X} \right\} \tag{1}$$

where r_f represents the log return on the riskless asset. In this section, for emphasis, we use bold notation to denote equilibrium returns.[1] The solution for the optimal fraction of wealth allocated to the risky asset is:

$$\theta^* = \frac{(\boldsymbol{\mu} - r_f)}{\gamma \sigma_v^2} X. \tag{2}$$

Thus, portfolio choice in a lognormal model with power utility resembles that of a mean variance investor. The allocation to the risky asset is increasing in the equilibrium average excess return, decreasing in risk aversion, and decreasing in the fundamental shock variance. Moreover, the fraction of wealth that an investor allocates to the risky asset strictly increases with expertise. The relationship is linear under our functional form assumptions.[2]

Equilibrium We now describe how the equilibrium excess return depends on the parameters for preferences, technology, and the joint distribution of wealth and expertise. We focus on comparative statics over the equilibrium average excess return, market level Sharpe ratio, and individual Sharpe ratios. We normalize the mass of investors to one, define the value of the supply of the risky asset to be S, determine the market clearing log expected return $\boldsymbol{\mu}$, and then back out the equilibrium expected level return and therefore $\boldsymbol{\alpha}$. We assume that W and X are jointly log-normally distributed. We denote the joint pdf of the log variables $f(w, x)$,

[1] Because an individual investor's return volatility depends on their expertise, for the approximation to be good given our specification for log return volatility, we have to impose a technical restriction that the majority of distribution of expertise X is bounded away from zero. This assumption is unnecessary if one adopts the general functional form for volatility discussed in footnote 1.1.

[2] Without restrictions on the distribution of X, θ can be larger than one, implying borrowing at the risk free rate.

with means and variances μ_w, μ_x, σ_w^2, and σ_x^2 respectively, and covariance $\rho_{w,x}\sigma_w\sigma_x$. Thus, an economy ψ is described by $\psi \equiv \{r_f, \gamma, I, \sigma_v, \mu_w, \sigma_w, \mu_x, \sigma_x, \rho_{w,x}\}$. The equilibrium log expected return $\boldsymbol{\mu}$ solves the market clearing condition:

$$\text{Supply} \equiv S = \text{Demand} = \int\int \exp(w)\theta^*\left(\exp(x)\right) f(w,x)\, dw\, dx = \frac{\boldsymbol{\mu} - r_f}{\gamma\sigma_v^2}\mathcal{X} \tag{3}$$

where $\theta^*\left(\exp(x)\right)$ is the portfolio choice given in Equation (2) and $\mathcal{X}$ is the wealth and population weighted average of expertise:

$$\int\int \exp(w+x) f(w,x)\, dw\, dx = \exp\left(\frac{1}{2}\left(\sigma_w^2 + \sigma_x^2 + 2\rho_{w,x}\sigma_w\sigma_x + 2\mu_w + 2\mu_x\right)\right) \tag{4}$$

utilizing the result for the expectation of log normally distributed variables.

Rearranging, we have:

$$\boldsymbol{\mu} - r_f = \left(\frac{\sigma_v^2}{\mathcal{X}}\right)\gamma S. \tag{5}$$

The equilibrium log expected excess return is increasing in the amount of risk relative to the risk bearing capacity of investors. We decompose the inputs into two components. The first term is the effective risk in the market, namely the fundamental risk σ_v^2, scaled down by the wealth and population weighted average of expertise. The second term is the risk aversion scaled supply of the risky asset which must be cleared. The higher is investors' coefficient of relative risk aversion, and the larger is the supply of the asset, the higher is the required return. Conversely, the wealth and population weighted average of expertise, $\mathcal{X}$, scales $\boldsymbol{\mu}$ down due to the positive impact of expertise on investors' allocation to the risky asset.

Using the equilibrium log expected return $\boldsymbol{\mu}$, we can rewrite agents' optimal portfolio allocations to the risky asset as:

$$\theta^* = \frac{X}{\mathcal{X}}S. \tag{6}$$

This expression captures the fact that, in equilibrium, the optimal portfolio allocations to the risky asset by an agent with expertise X turns out to be a fraction of total supply equal to their expertise relative to the wealth and population weighted average of expertise.

The equilibrium mean of the level of the gross risky return over the level of the gross risk free rate, $\boldsymbol{\alpha}$, is a monotonic transformation of $\boldsymbol{\mu}$. In particular, we show in the Appendix that the equilibrium α is then given by:

$$\boldsymbol{\alpha} = \exp\left(\boldsymbol{\mu}\right) - R_f \tag{7}$$

which gives a one to one mapping from μ to α conditional on parameters. Note also that writing θ^* (Equation 2) as a function of either $\boldsymbol{\mu}$ or $\boldsymbol{\alpha}$ will always yield identical equilibrium outcomes.

Lemma 1.1 *Using Equation (7) describing the equilibrium market clearing $\boldsymbol{\alpha}$, the following comparative statics can be directly calculated:*

1. $\frac{\partial \boldsymbol{\alpha}}{\partial \sigma_v^2} = \exp(\boldsymbol{\mu}) \frac{\gamma S}{\mathcal{X}} > 0$. *$\boldsymbol{\alpha}$ increases with fundamental risk.*

2. $\frac{\partial \boldsymbol{\alpha}}{\partial \gamma} = \exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} S > 0$. *$\boldsymbol{\alpha}$ increases with the coefficient of relative risk aversion.*

3. $\frac{\partial \boldsymbol{\alpha}}{\partial S} = \exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma > 0$. *$\boldsymbol{\alpha}$ increases with the risky asset supply investors must absorb.*

4. $\frac{\partial \boldsymbol{\alpha}}{\partial \mu_w} = -\exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma S < 0$. *$\boldsymbol{\alpha}$ decreases as aggregate wealth increases.*

5. $\frac{\partial \boldsymbol{\alpha}}{\partial \mu_x} = -\exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma S < 0$. *$\boldsymbol{\alpha}$ decreases as aggregate expertise increases.*

6. $\frac{\partial \boldsymbol{\alpha}}{\partial \rho_{w,x}} = -\exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma S \sigma_w \sigma_x < 0$. *As $\rho_{w,x}$ increases, there is a more efficient allocation of expertise and $\boldsymbol{\alpha}$ decreases.*

7. $\frac{\partial \boldsymbol{\alpha}}{\partial \sigma_w} = -\exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma S (\sigma_w + \rho_{w,x}\sigma_x)$
   - > 0 *if* $\rho_{w,x} < -\frac{\sigma_w}{\sigma_x}$, *i.e. if wealth and expertise are strongly negatively correlated.*
   - < 0 *if* $\rho_{w,x} > -\frac{\sigma_w}{\sigma_x}$, *i.e. if wealth and expertise are positively or only weakly negatively correlated.*

8. $\frac{\partial \boldsymbol{\alpha}}{\partial \sigma_x} = -\exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma S (\sigma_x + \rho_{w,x}\sigma_w)$
   - > 0 *if* $\rho_{w,x} < -\frac{\sigma_x}{\sigma_w}$, *i.e. if wealth and expertise are strongly negatively correlated.*
   - < 0 *if* $\rho_{w,x} > -\frac{\sigma_x}{\sigma_w}$, *i.e. if wealth and expertise are positively or only weakly negatively correlated.*

Proof. By direct calculation. ■

All comparative statics are intuitive. An increase in the correlation of wealth and expertise will reduce $\boldsymbol{\alpha}$, as investors with more expertise account for a larger share of the wealth distribution. The effect of an increase in $\rho_{w,x}$ on the market clearing $\boldsymbol{\alpha}$ will be larger the larger is amount of fundamental risk, σ_v^2, the larger is the coefficient of relative risk aversion, γ, the larger is the supply of the risky asset, S, the smaller is the mean of log wealth, μ_w, and the smaller is the mean of log expertise, μ_x.

We also derive results for the equilibrium market-level and investor-specific Sharpe ratios. The market level Sharpe ratio requires a definition appropriate for our environment. Here, we define the equilibrium market level Sharpe ratio to be the equal-weighted cross-sectional average of excess returns divided by the equal-weighted cross-sectional standard deviation. Thus this market Sharpe ratio can, for example, be interpreted as the expected Sharpe ratio for an investor "behind the veil" drawing from the distribution of possible levels of expertise, before the investment stage of the model. This Sharpe ratio would be relevant, for example, in a model with entry in which an investor must decide whether to enter before drawing an expertise level from the given distribution. We then refer to what is technically the equilibrium equally weighted market Sharpe ratio as the "Sharpe ratio" for exposition purpose:

$$SR = \frac{1 - R_f \ \exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right] - 1}} = \frac{1 - R_f \exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\sum_{k=1}^{\infty} \frac{1}{k!}\sigma_v^{2k} \exp(-k\mu_x + \frac{1}{2}k^2\sigma_x^2)}} \approx \frac{1 - R_f \exp\left(-\boldsymbol{\mu}\right)}{\sigma_v \exp(-\frac{1}{2}\mu_x + \frac{1}{4}\sigma_x^2)}, \quad (8)$$

where $\mathbb{E}$ denotes the cross-sectional expectation. This ratio aggregates all investor decisions and measures the *market level* risk return tradeoff.[3] The market-level Sharpe ratio increases as the average log expertise μ_x in this economy increases, but it decreases as the cross-sectional standard deviation of log expertise σ_x^2 increases.

Lemma 1.2 *Using Equation (8) describing the equilibrium market clearing equally weighted Sharpe ratio, the following comparative statics can be directly calculated:*

1. *Let η denote any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$.*
   Then, $Sign\left(\frac{\partial(SR)}{\partial\eta}\right) = Sign\left(\frac{\partial\boldsymbol{\alpha}}{\partial\eta}\right)$.
2. *The signs for comparative statics with respect to parameters $\hat{\eta} \in \{\sigma_v^2, \mu_x, \sigma_x\}$, are indeterminate.*

Proof. By direct calculation, see Appendix. ■

Expected returns rise proportionally relative to the volatility of the risky asset return in our static model, so that the Sharpe ratio improves with any parameter change that increases $\boldsymbol{\alpha}$. Thus, we confirm that, at the market level, parameter changes which lead to an increase

[3]See Appendix for derivation. We also compute and analyze the market value weighted Sharpe ratio in the Appendix.

in the equilibrium expected excess return in fact lead to better investment opportunities given the market risk in equilibrium.

In our model, each investor confronts a different risk-return trade-off. Since the volatility of log returns depends on individual investors' expertise, an observed increase in the market Sharpe ratio does not necessarily imply a higher Sharpe ratio for every investor in the market. Moreover, even if the Sharpe ratio improves for each agent individually, the magnitude of the improvement an individual investor faces will not, in general, coincide with the market improvement. To see this, consider the investor-specific Sharpe ratio. For an investor with wealth W and expertise X, we show in the Appendix that this investor's Sharpe ratio is given by:

$$SR\,(X) = \frac{1 - R_f\ \exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}. \tag{9}$$

Equation (9) clearly shows that the model can deliver considerable cross-sectional dispersion in investor-specific Sharpe ratios. Investors with very low effective risk, $\frac{\sigma_v^2}{X}$, face significantly higher Sharpe ratios than their counterparts with low expertise. We can determine the signs of the following comparative statics:

Lemma 1.3 *Using Equation (9) describing the investor-specific Sharpe ratio, and Equation (2) describing the portfolio allocation θ^*, the following comparative statics can be directly calculated. Let η denote any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$.*[4]

1. $\frac{\partial SR(X)}{\partial X} > 0$. *Higher expertise generates lower effective risk, and a correspondingly higher individual Sharpe ratio.*

2. $Sign\left(\frac{\partial SR(X)}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right)$. *All investor-specific Sharpe ratios co-move with the equilibrium excess return and the market level equilibrium Sharpe ratio.*

3. $Sign\left(\frac{\partial \boldsymbol{Var}(SR(X))}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right)$. *Whenever a parameter change increases the market level equilibrium Sharpe ratio, it leads to a larger cross-sectional dispersion in the investor-specific Sharpe ratio.*

[4]Derivatives with respect to μ_x and σ_x follow the same formulas as those that support parts 2 to 5 of lemma 1.3. However, the changes are not comparable to the market Sharpe ratio, as we can't determine the signs in lemma 1.2, part 2. Derivatives with respect to σ_v^2 cannot be signed generally.

4. $Sign\left(\frac{\partial^2 SR(X)}{\partial\eta\partial X}\right) = Sign\left(\frac{\partial SR(X)}{\partial\eta}\right) = Sign\left(\frac{\partial\boldsymbol{\alpha}}{\partial\eta}\right) = Sign\left(\frac{\partial(SR)}{\partial\eta}\right)$.
Whenever a parameter change increases the investor-specific Sharpe ratio, it leads to a larger increase for high expertise investors relative to low expertise investors.

5. $Sign\left(\frac{\partial^2 \theta^*(X)}{\partial\eta\partial X}\right) = Sign\left(\frac{\partial SR(X)}{\partial\eta}\right) = Sign\left(\frac{\partial\boldsymbol{\alpha}}{\partial\eta}\right) = Sign\left(\frac{\partial(SR)}{\partial\eta}\right)$.[5]
Whenever a parameter change increases the investor-specific portfolio allocation, it leads to a larger increase for high expertise investors relative to low expertise investors.

6. $\exists \bar{X} > \underline{X} > 0$ *such that* $\forall X > \bar{X}, \frac{\partial SR(X)}{\partial\sigma_v^2} > 0$, *and* $\forall X < \underline{X}, \frac{\partial SR(X)}{\partial\sigma_v^2} < 0$. *An increase in the fundamental risk generates a higher Sharpe ratio for high expertise investors and a lower Sharpe ratio for low expertise investors.*

Proof. By direct calculation. See Appendix. ■

Lemma (1.3) has rich implications. First, we emphasize the co-movement between cross sectional variation in investor-specific Sharpe ratios and the level of the market Sharpe ratio. Any increase in the market-level Sharpe ratio will also increase the cross-sectional dispersion in Sharpe ratios. Furthermore, because an increase in the market level Sharpe ratio improves investment opportunities for high expertise investors by more than for low expertise investors, such an increase accordingly increases their allocation to the risky asset θ^* by more. Thus, an improvement in the market-level risk return tradeoff in large part reflects the improved risk-return trade-off faced by high expertise investors, and not by their low expertise counterparts.

Any parameter change which increases the market level Sharpe ratio increases the investor specific Sharpe ratio for high expertise by more, and increases the influence of high expertise investors' Sharpe ratios on the market level risk return tradeoff. In our model, measured improvements in the aggregate Sharpe ratio are a misleading indicator of improvements in individual investors' risk-return tradeoff, and can indeed more accurately reflect changes in the Sharpe ratio of higher expertise investors. The converse is also true.

Furthermore, part 6 of Lemma (1.3) states that changes to fundamental risk can lead to changes in individual Sharpe ratios that vary in sign. For example, if σ_v^2 increases, all investors face the same increase in the equilibrium excess return, but investors with high expertise face a considerably smaller increase in risk. Thus, more complex assets with higher σ_v^2 can have higher market level Sharpe ratios but lower demand from non-experts. We also emphasize that

[5] Except for γ, where $\frac{\partial^2\theta^*(X)}{\partial\gamma\partial X} = 0$.

because an increase in fundamental risk improves the investor-specific Sharpe ratio for some investors but not others, in a dynamic model a shock to fundamental risk can lead potentially lead to variation in investors' participation decisions. In other words, an increase in risk which improves the market level equally weighted Sharpe ratio may still lead low expertise investors to exit, or not to enter.

2 Proofs for Static Model Results

This section contains proofs and additional results for the static model.

Optimal Portfolio Choice

This section describes how to solve the optimal portfolio allocation problem in the static model. We also use upper case letters for level variables, and lower case letters for log variables. Under the assumptions in the main text, the optimization problem for an investor with wealth W and expertise X, can be written as:

$$v(W,X) = \max_{\theta} \mathbb{E}\left[\frac{(WR_p)^{1-\gamma}}{1-\gamma}\right]$$

subject to

$$\begin{aligned} R_p &= \theta R + (1-\theta) R_f, \\ r|(W,X) &\sim N\left(\boldsymbol{\mu} - \frac{1}{2}\frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right). \end{aligned}$$

? and **?** show that the log portfolio return r_p over a short time horizon with bounded variance, can be approximated by:

$$r_p \approx r_f + \theta(r - r_f) + \frac{1}{2}\theta(1-\theta)\frac{\sigma_v^2}{X}.$$

As a result,

$$r_p|(W,X) \sim N\left(r_f + \theta(\boldsymbol{\mu} - r_f) - \frac{1}{2}\theta^2\frac{\sigma_v^2}{X}, \theta^2\frac{\sigma_v^2}{X}\right).$$

Then the value function equals:

$$v(W,X) = \max_{\theta} \frac{W^{1-\gamma}}{1-\gamma} \exp\left((1-\gamma)\, r_f + (1-\gamma)\,\theta\,(\boldsymbol{\mu} - r_f) - \frac{1}{2}\gamma\,(1-\gamma)\,\theta^2 \frac{\sigma_v^2}{X} \right).$$

Hence, The investor's optimization problem becomes:

$$\max_{\theta} \left\{ \theta\,(\boldsymbol{\mu} - r_f) - \frac{\gamma}{2}\theta^2 \frac{\sigma_v^2}{X} \right\}.$$

Equilibrium Market Excess Return

This section describes how to derive the equilibrium market excess return, $\boldsymbol{\alpha}$, from the log expected return, $\boldsymbol{\mu}$, given all parameters. Because

$$r|\left(W,X\right)\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_v^2}{X},\frac{\sigma_v^2}{X}\right),$$

Then

$$\mathbb{E}\left(R|W,X\right)=\exp\left(\boldsymbol{\mu}\right).$$

In addition,

$$\mathbb{E}\left[R\right]=\mathbb{E}\left[\mathbb{E}\left(R|W,X\right)\right].$$

Hence,

$$\mathbb{E}\left[R\right]=\exp\left(\boldsymbol{\mu}\right).$$

Finally,

$$\alpha=\exp\left(\boldsymbol{\mu}\right)-R_f.$$

Equilibrium Equally Weighted Market Sharpe ratio

This section describes how to derive the equilibrium equally weighted market Sharpe ratio, SR, from the log expected return, $\boldsymbol{\mu}$, given all parameters. Because

$$r|\left(W,X\right)\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_{\upsilon}^{2}}{X},\frac{\sigma_{\upsilon}^{2}}{X}\right),$$

Then

$$\boldsymbol{Var}\left(R|W,X\right)=\exp\left(2\boldsymbol{\mu}\right)\left(\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)-1\right).$$

In addition, we have proven that

$$\mathbb{E}\left(R|W,X\right)=\mathbb{E}\left[R\right]=\exp\left(\boldsymbol{\mu}\right).$$

Therefore, the equally weighted variance of the risky asset, is given by:

$$\boldsymbol{Var}\left[R\right]=\mathbb{E}\left(\boldsymbol{Var}\left(R|W,X\right)\right)=\exp\left(2\boldsymbol{\mu}\right)\left(\mathbb{E}\left[\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)\right]-1\right).$$

Hence, the equally weighted market Sharpe ratio, can be written as:

$$SR=\frac{1-R_{f}\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\mathbb{E}\left[\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)\right]-1}},$$

where $\mathbb{E}\left[\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)\right]=\sum_{k=0}^{\infty}\frac{1}{k!}\sigma_{\upsilon}^{2k}\mathbb{E}[\exp(-kx)]$, using a Taylor expansion of $\exp(\sigma_{\upsilon}^{2}X^{-1})=1+\sigma_{\upsilon}^{2}X^{-1}+\frac{1}{2!}\sigma_{\upsilon}^{4}X^{-2}+\frac{1}{3!}\sigma_{\upsilon}^{6}X^{-3}+\ldots$, which is equivalent to:

$$\mathbb{E}\left[\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)\right]=\sum_{k=0}^{\infty}\frac{1}{k!}\sigma_{\upsilon}^{2k}[\exp(-k\mu_{x}+\frac{1}{2}k^{2}\sigma_{x}^{2})],$$

where we have used the moment-generating function of the normal distribution. Hence, the equally weighted market Sharpe ratio, can be written as:

$$SR=\frac{1-R_{f}\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\sum_{k=0}^{\infty}\frac{1}{k!}\sigma_{\upsilon}^{2k}\exp(-k\mu_{x}+\frac{1}{2}k^{2}\sigma_{x}^{2})-1}},$$

Provided that σ_v^4 is small enough, this SR is approximately equal to the following expression:

$$SR \approx \frac{1 - R_f \exp\left(-\boldsymbol{\mu}\right)}{\sigma_v \exp(-\frac{1}{2}\mu_x + \frac{1}{4}\sigma_x^2)}.$$

Equilibrium Investor-Specific Sharpe ratio

This section describes how to derive the equilibrium investor-specific Sharpe ratio, $SR(X)$, from log expected return, $\boldsymbol{\mu}$, given all parameters. For an investor with wealth W and expertise X, Because

$$r|\left(W,X\right)\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_{v}^{2}}{X},\frac{\sigma_{v}^{2}}{X}\right),$$

Then

$$\mathbb{E}\left(R|W,X\right)=\exp\left(\boldsymbol{\mu}\right),$$

And

$$\boldsymbol{Var}\left(R|W,X\right)=\exp\left(2\boldsymbol{\mu}\right)\left(\exp\left(\frac{\sigma_{v}^{2}}{X}\right)-1\right).$$

Hence, the investor-specific Sharpe ratio is given by:

$$SR\left(X\right)=\frac{1-R_{f}\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\exp\left(\frac{\sigma_{v}^{2}}{X}\right)-1}}$$

$$E\left[SR\left(X\right)\right]=E\left[\frac{1-R_{f}\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\exp\left(\frac{\sigma_{v}^{2}}{X}\right)-1}}\right]$$

Proof of Lemma 1.2 and 1.3

This section describes how to prove lemma 1.2 and 1.3. From equations (2), (7), (8) and (9), we can derive that, if η denotes any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$:

1. $\frac{\partial \boldsymbol{\alpha}}{\partial \eta} = \exp(\boldsymbol{\mu}) \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
2. $\frac{\partial (SR)}{\partial \eta} = \frac{R_f \exp(-\boldsymbol{\mu})}{\sqrt{\mathbb{E}\left(\exp\left(\frac{\sigma_v^2}{X}\right)\right) - 1}} \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
3. $\frac{\partial SR(X)}{\partial \eta} = \frac{R_f \exp(-\boldsymbol{\mu})}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}} \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
4. $\frac{\partial \boldsymbol{Var}(SR(X))}{\partial \eta} = 2\left(1 - R_f \exp(-\boldsymbol{\mu})\right) R_f \exp(-\boldsymbol{\mu}) \boldsymbol{Var}\left(\frac{1}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}\right) \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
5. $\frac{\partial^2 SR(X)}{\partial \eta \partial X} = \frac{\partial\left(\frac{R_f \exp(-\boldsymbol{\mu})}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}\right)}{\partial\left(\frac{\sigma_v^2}{X}\right)} \frac{\partial\left(\frac{\sigma_v^2}{X}\right)}{\partial X} \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
6. $\frac{\partial^2 \theta^*(X)}{\partial \eta \partial X} = \frac{1}{\gamma \sigma_v^2} \frac{\partial \boldsymbol{\mu}}{\partial \eta}, \forall \eta \neq \gamma$, and $\frac{\partial \theta^*(X)}{\partial \gamma} = 0$
7. $\frac{\partial SR(X)}{\partial \sigma_v^2} = \frac{R_f \exp(-\boldsymbol{\mu}) \frac{\boldsymbol{\mu} - r_f}{\sigma_v^2} - \frac{1}{2}\left(1 - R_f \exp(-\boldsymbol{\mu})\right) \frac{\exp\left(\frac{\sigma_v^2}{X}\right) \frac{1}{X}}{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}$.

Hence, $Sign\left(\frac{\partial \boldsymbol{\mu}}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right) = Sign\left(\frac{\partial SR(X)}{\partial \eta}\right)$
$= Sign\left(\frac{\partial \boldsymbol{Var}(SR(X))}{\partial \eta}\right) = Sign\left(\frac{\partial^2 SR(X)}{\partial \eta \partial X}\right) = Sign\left(\frac{\partial^2 \theta^*(X)}{\partial \eta \partial X}\right)$

In addition, we have:

1. Because $\frac{\exp\left(\frac{\sigma_v^2}{X}\right)\frac{1}{X}}{\exp\left(\frac{\sigma_v^2}{X}\right)-1} > \frac{1}{X}$, $\forall X$,

   then $\frac{\partial SR(X)}{\partial \sigma_v^2} < \frac{R_f \exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2} - \frac{1}{2}\left(1-R_f\exp(-\boldsymbol{\mu})\right)\frac{1}{X}}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right)-1}} < 0$, $\forall X < \underline{X}$,

   where $\underline{X} = \frac{\frac{1}{2}\left(1-R_f\exp(-\boldsymbol{\mu})\right)}{R_f\exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2}} > 0$;

2. $0 = \frac{1}{\sigma_v^2}\left(1+\frac{\sigma_v^2}{X}\right) - \frac{1}{X} - \frac{1}{\sigma_v^2} < \frac{1}{\sigma_v^2}\exp\left(\frac{\sigma_v^2}{X}\right) - \frac{1}{X} - \frac{1}{\sigma_v^2} = \left(\exp\left(\frac{\sigma_v^2}{X}\right)-1\right)\left(\frac{1}{X}+\frac{1}{\sigma_v^2}\right) - \exp\left(\frac{\sigma_v^2}{X}\right)\frac{1}{X}$,

   then $\frac{\exp\left(\frac{\sigma_v^2}{X}\right)\frac{1}{X}}{\exp\left(\frac{\sigma_v^2}{X}\right)-1} < \frac{1}{X}+\frac{1}{\sigma_v^2}$,

   and $\frac{\partial SR(X)}{\partial \sigma_v^2} > \frac{R_f\exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2} - \frac{1}{2}\left(1-R_f\exp(-\boldsymbol{\mu})\right)\left(\frac{1}{X}+\frac{1}{\sigma_v^2}\right)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right)-1}}$, $\forall X$.

   Hence, if $\bar{X} = \frac{1}{\frac{R_f\exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2}}{\frac{1}{2}\left(1-R_f\exp(-\boldsymbol{\mu})\right)} - \frac{1}{\sigma_v^2}} > 0$, then $\forall X > \bar{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} > 0$.

   $\bar{X} > 0$ if and only if $F(\boldsymbol{\mu} - r_f) \equiv \exp\left(-\left(\boldsymbol{\mu}-r_f\right)\right)\left(\boldsymbol{\mu}-r_f+\frac{1}{2}\right) - \frac{1}{2} > 0$.

   We can prove $F(\boldsymbol{\mu}-r_f) > 0$ if and only if $0 < \boldsymbol{\mu} - r_f < 1.2564$, but $\boldsymbol{\mu}-r_f = 1.2564$ corresponds to an $\boldsymbol{\alpha}$ around 250%. Then we can conclude $\bar{X} > 0$ for all reasonable parameters.

3. we can prove by direct computation that $\bar{X} > \underline{X}$ whenever $\bar{X} > 0$.

In sum, for all reasonable parameters, $\exists \bar{X} > \underline{X} > 0$ such that $\forall X > \bar{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} > 0$, and $\forall X < \underline{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} < 0$. The general functional form for effective risk yields similar results.

Equilibrium Value-Weighted Market Sharpe ratio

This section shows that our main conclusions still hold with respect to the value-weighted equilibrium market Sharpe ratio. Because

$$r|\left(W,X\right)\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_{\upsilon}^{2}}{X},\frac{\sigma_{\upsilon}^{2}}{X}\right),$$

Then

$$\mathbb{E}\left(R|W,X\right)=\exp\left(\boldsymbol{\mu}\right).$$

Then the value-weighted market expected return also equals to:

$$\exp\left(\boldsymbol{\mu}\right),$$

In addition,

$$\boldsymbol{Var}\left(R|W,X\right)=\exp\left(2\boldsymbol{\mu}\right)\left(\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)-1\right).$$

Therefore, the value-weighted variance of the risky asset, is given by:

$$\int\int\boldsymbol{Var}\left(R|W,X\right)\frac{\exp(w)\theta^{*}\left(\exp(x)\right)f(w,x)}{\int\int\exp(w)\theta^{*}\left(\exp(x)\right)f(w,x)\,dw\,de}\,dw\,de,$$

Which equals to

$$\exp\left(2\boldsymbol{\mu}\right)\frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)-1\right)\exp(w+x)\right]}{\mathcal{X}}$$

Hence, the value-weighted market Sharpe ratio, can be written as:

$$\frac{1-R_{f}e^{-\mu}}{\sqrt{\frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)-1\right)\exp(w+x)\right]}{\mathcal{X}}}}.$$

where $\mathbb{E}\left[\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}+w+x\right)\right]$, using a Taylor expansion of $\exp(\sigma_{\upsilon}^{2}X^{-1}+w+x)=1+\sigma_{\upsilon}^{2}X^{-1}+w+x+\frac{1}{2!}(\sigma_{\upsilon}^{2}X^{-1}+w+x)^{2}+\frac{1}{3!}(\sigma_{\upsilon}^{2}X^{-1}+w+x)^{3}+\ldots$, which is equivalent to:

$$\mathbb{E}\left[\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}+w+x\right)\right]=\sum_{k=0}^{\infty}\frac{1}{k!}(\sigma_{\upsilon}^{2}X^{-1}+w+x)^{k})],$$

This will be approximately equal to

$$\begin{aligned} E[\exp(\sigma_v^2 X^{-1} + w + x)] \quad & \approx 1 + E[\sigma_v^2 X^{-1}] + E[w] + E[x], \\ & + \frac{1}{2} E[\sigma_v^4 X^{-2} + w^2 + x^2 + 2wx\sigma_v^4 X^{-2} + 2w^2 x\sigma_v^2 X^{-1} + 2wx^2\sigma_v^2 X^{-1}] \end{aligned}$$

The moment-generating function is given by:

$$M(t_1, t_2) = E[\exp(t_1 w)\exp(t_2 x)] = \exp\left(t_1\mu_x + t_2\mu_w + (1/2)(t_1\sigma_x^2 + t_2\sigma_w^2 + 2t_1 t_2 wx\rho_{w,x}\sigma_x\sigma_w)\right)$$

$\partial M(t_1, t_2)$

Then, if η denotes any parameter $\eta \in \{\gamma, S\}$,

$$\frac{\delta\left(SR\right)}{\delta\eta} = \frac{R_f e^{-\mu}}{\sqrt{\frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_v^2}{X}\right)-1\right)\exp(w+x)\right]}{\mathcal{X}}}} \frac{\delta\mu}{\delta\eta},$$

However, unlike in the case of the equally weighted market equilibrium Sharpe ratio, for the value weighted Sharpe ratio, the derivatives needed to sign the comparative statics in lemma 1.2 and 1.3 for $\eta \in \{\mu_w, \sigma_w, \rho_{w,x}\}$ are indeterminate.

Wealth effect of Expertise

This section shows that while savings rates can theoretically be slightly decreasing in expertise, due to the wealth effect from higher expertise and the associated larger present value of investment opportunities, this effect tends to be dominated by the portfolio choice effect.

The static model with a consumption savings decision can be written as:

$$v(W,X) = \max_{(I,\theta)} \frac{(W-I)^{1-\gamma}}{1-\gamma} + \beta I^{1-\gamma}\mathbb{E}\left[\frac{R_p^{1-\gamma}}{1-\gamma}\right]$$

subject to:

$$\begin{aligned} R_p &= \theta R + (1-\theta) R_f, \\ r|\left(W,X\right) &\sim N\left(\boldsymbol{\mu} - \frac{1}{2}\frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right). \end{aligned}$$

Clearly, the portfolio choice problem is independent from the consumption savings decision, and the solution to the portfolio choice problem coincides with that of the static model without the consumption saving decision. For any choice of investment I, the optimal portfolio allocation always solves the same problem, maximizing the expected utility derived from the chosen investment level, given the return process for the riskless and risky assets. Therefore, we can plug the optimal portfolio choice back into the value function, and then derive the optimal investment. Finally we get:

$$I^* = W\frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}$$

where

$$E\left[R_p^{1-\gamma}\right] = \exp\left((1-\gamma)\, r_f + \frac{1}{2}\frac{(1-\gamma)}{\gamma}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}\right).$$

Then, we can show that:

$$\frac{\partial I^*}{\partial X} = W\frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{\left(1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}\right)^2}\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\left(\frac{\sigma_v^2}{X}\right)^2}\frac{\partial\left(\frac{\sigma_v^2}{X}\right)}{\partial X}.$$

Observe that the saving rate decreases with the expertise if and only if $\gamma > 1$.

However, for the investment in the risky asset, $I^*\theta^*$, we have:

$$I^*\theta^* = W \frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}} \frac{(\boldsymbol{\mu}-r_f)}{\gamma \frac{\sigma_v^2}{X}}.$$

Then,

$$\frac{\partial\left(I^*\theta^*\right)}{\partial X} = W \frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}} \frac{(\boldsymbol{\mu}-r_f)}{\gamma\left(\frac{\sigma_v^2}{X}\right)^2} \left(\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}} \frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1\right) \frac{\partial\left(\frac{\sigma_v^2}{X}\right)}{\partial X}$$

There are two cases, depending on the coefficient of relative risk aversion:

1. If $\gamma < 1$, the saving rate does not fall with the expertise, neither does the investment in the risky asset.
   We have $\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{\left(\boldsymbol{\mu}-r_f\right)^2}{\frac{\sigma_v^2}{X}} \frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1<0$.
   Therefore, $\frac{\partial(I^*\theta^*)}{\partial X} > 0, \forall X$.

2. If $\gamma > 1$, the saving rate falls with the expertise, while the investment in the risky asset doesn't, as long as the expertise level is not too high.
   We have $\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{\left(\boldsymbol{\mu}-r_f\right)^2}{\frac{\sigma_v^2}{X}} \frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1 < \frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{\left(\boldsymbol{\mu}-r_f\right)^2}{\frac{\sigma_v^2}{X}}-1, \forall X$.
   Then $\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{\left(\boldsymbol{\mu}-r_f\right)^2}{\frac{\sigma_v^2}{X}} \frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1<0, \forall X<\bar{X}$, where $\bar{X} = \frac{2\gamma^2}{(\gamma-1)}\frac{\sigma_v^2}{\left(\boldsymbol{\mu}-r_f\right)^2}$.
   Therefore, $\frac{\partial(I^*\theta^*)}{\partial X} > 0, \forall X < \bar{X}$. The signs for comparative statics for $\forall X > \bar{X}$ are indeterminate.

In sum, investment in the risky asset increases with expertise, as long as the expertise level is not too high. The general functional form for effective risk yields similar results.