

Complex Asset Markets*

Andrea L. Eisfeldt[†]

UCLA Anderson

Hanno Lustig[‡]

Stanford GSB

Lei Zhang[§]

HKU

October 10, 2019

Abstract

Complex assets appear to have high average returns and high Sharpe ratios. However, despite free entry, participation in complex assets markets is limited, because investing in complex assets requires a model, and investors' individual models expose them to investor-specific risk. Investors with higher expertise have better models, and thus face lower risk. We construct a dynamic economy in which the joint distribution of wealth and expertise determines aggregate risk bearing capacity in the long-run equilibrium. In this equilibrium, more complex asset markets, i.e. those which are more difficult to model, have *lower* participation rates, despite having *higher* market-level Sharpe ratios, provided that asset complexity and expertise are complementary. We provide evidence of idiosyncratic risk in complex asset markets, and show that a positive shock to the quantity of idiosyncratic risk induces exit by less expert incumbents, in spite of higher alphas, leading to an increase in the market-level Sharpe ratio. This selection mechanism helps to explain the dynamics of returns, volatility, and participation in MBS markets during the financial crisis.

Key Words: heterogeneous agent models, industry equilibrium, firm size distribution, segmented markets, limits of arbitrage, selection.

*We thank seminar participants at Wharton, UNC, OSU, the Minnesota Macro Asset Pricing Conference, the Eighth Annual Conference of the Paul Woolley Centre (PWC) for the Study of Capital Market Dysfunctionality, BYU, UCLA Anderson, Wash U., the AFA, the NBER Conference on Long-Term Asset Management, UCSD, University of Michigan, Stanford, CREI, Bocconi, LSE, UCL, and University of Washington, as well as Santiago Bzdresch, Philip Bond, Sebastian DiTella, Vincent Glode, Peter Kondor, Arvind Krishnamurthy, Pablo Kurlat, Leo Li, Francis Longstaff, Benjamin Moll, Tyler Muir, Dimitris Papanikolaou, Scott Richard, Alexi Savov, Alexis Toda, and Dimitry Vayanos for helpful comments and discussions. We also thank Leo Li for exceptional research assistance.

[†]Finance Area, Anderson School of Management, UCLA, and NBER. andrea.eisfeldt@anderson.ucla.edu

[‡]Finance Area, Stanford GSB, and NBER. hlustig@stanford.edu

[§]Faculty of Business and Economics, the University of Hong Kong. econlei@hku.hk

1 Introduction

Investing in complex assets, such as Mortgage-Backed Securities (MBS), Convertible Bonds, Collateralized Loan Obligations, and dynamic long-short equity portfolios, requires pricing and hedging models. These models are investor-specific, exposing investors to idiosyncratic risk. If managers must retain significant exposures to their own portfolios, then idiosyncratic risk leads to “alpha”, even in a long-run equilibrium. One puzzle, given their attractive risk-return tradeoffs, is why complex assets do not attract broad participation. Instead, most complex assets only attract sophisticated investors, such as hedge funds. We argue that complex assets display higher average returns, higher Sharpe ratios, and lower participation because of variation amongst investors in the quality of their pricing and hedging models. Investors with greater expertise have superior models. Superior models allow expert investors to earn alpha in complex asset markets with lower exposures to idiosyncratic model risk than their less sophisticated counterparts. The risk-bearing capacity of expert investors drives equilibrium alpha below the level required for inexperienced investors. Inferior models expose less sophisticated investors to greater risk per unit of expected return, thereby endogenously limiting participation.

Mortgage Backed Securities (MBS) are a good example of a complex asset class. Active investors develop proprietary prepayment models which they use to construct the Option Adjusted Spreads (OAS) used for pricing securities and hedging interest rate risk. Figure 1 plots the median percentage dispersion in OAS across six major dealers. Clearly, these six dealers’ models exhibit considerable heterogeneity, with the variation in spreads across dealers often exceeding the level of the spread itself. Moreover, model differences appear to be amplified during and after the financial crisis. Most of our analysis is cross sectional, focusing on why more complex assets exhibit higher alphas, higher Sharpe ratios, and lower participation. Consistent with MBS being complex assets, Duarte, Longstaff, and Yu (2006) show that MBS earn the highest average returns, and Sharpe (1966) ratios, out of five fixed income arbitrage strategies they study.¹ Consistent with our model, they argue that the superior risk-return tradeoff of MBS is due to the significant intellectual capital required to implement a successful MBS investment strategy.

Consistent with the considerable heterogeneity in OAS across the major dealers, the portfolios of hedge fund investors that are active in MBS markets (1) display considerable idiosyncratic volatility and (2) experienced a large increase in idiosyncratic volatility during the financial crisis. Table 1 reports the annualized volatility of fund-specific returns in the MBS category of Hedge Funds in the Thomson-Reuters Database of Hedge Funds. We develop two measures

¹Gabaix, Krishnamurthy, and Vigneron (2007) and Diep, Eisfeldt, and Richardson (2017) provide evidence that MBS returns are driven in large part by market segmentation.

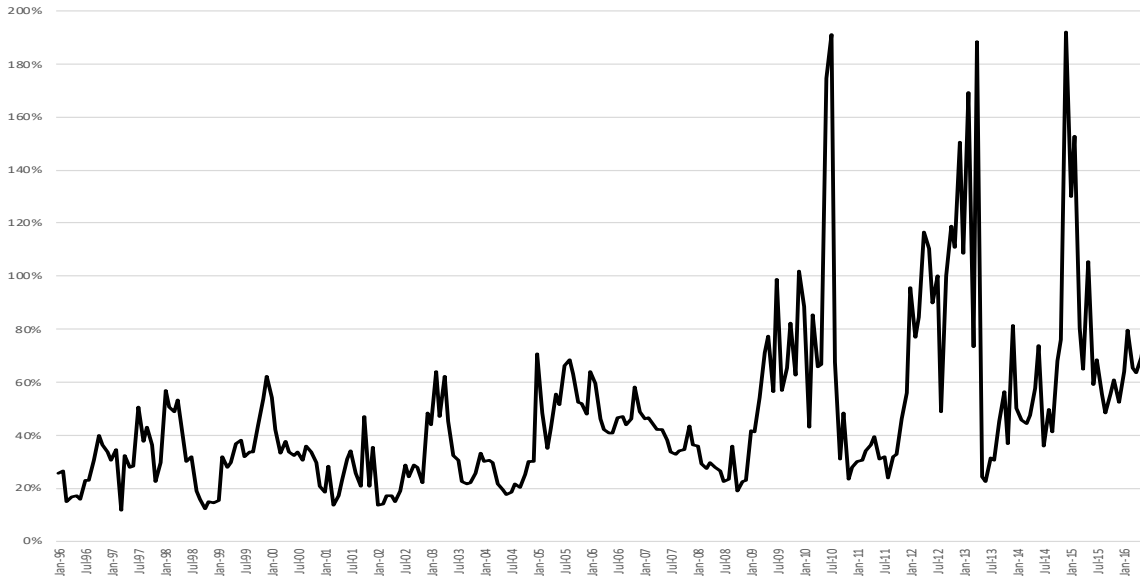


Figure 1: This figure plots the median percentage dispersion across six major dealers' Option Adjusted Spreads (OAS) on Fannie Mae 30 year fixed-rate Mortgage Backed Securities. Each month, the cross-sectional standard deviation in OAS across dealers is computed for each available coupon, and normalized by the median OAS for that coupon. The plotted series is the median percentage dispersion across coupons for each month.

of fund-specific volatility, and earn positive average returns. The first measure reports the (time-series average of the) cross-sectional standard deviation of returns. The second measure reports the (cross-sectional average of the) time-series standard deviation of the residuals in a regression of returns on the value-weighted returns. The second measure is more conservative, because the cross-sectional standard deviation is sensitive to cross-sectional differences in exposures to the common factor. We focus our discussion on the sample of 160 MBS hedge funds that are single-strategy funds (Panel A), however results are similar for single-strategy funds with a US focus only, and if we include multi-strategy funds (Panels B and C). The first column shows that MBS funds display considerable idiosyncratic volatility. The next three columns show that during the financial crisis (2007.6-2009.6), the fund-specific cross-sectional volatility (time-series volatility) of MBS funds increases from 5.66% (3.76%) at the start of the crisis to 20.14% (11.21%) at the end of the crisis. At the same, there is a large decline in realized equal-weighted excess returns from 9.88% before the crisis to -6.17% during the crisis. After the crisis, returns recover to 7.29% per annum. Regardless of the exact measure, we find that fund-specific volatility increased by close to 300% during the crisis.

This increase in idiosyncratic volatility was accompanied by large declines in realized returns during the crisis, and significant losses of wealth, accompanied by fund exit. Table 2 reports

Table 1: MBS Hedge Fund Returns. Annualized Returns in Asset-Backed Fixed Income. Sample: Monthly data from 1991.12-2018.08. The crisis is defined as the period from 2007.6-2009.6. We report the averages for the pre-crisis, crisis, and post-crisis periods, as well as the 12-month average prior to the start and the end of the crisis. XS Vol reports the time-series average of the cross-sectional standard deviation of returns. TS Vol reports the cross-sectional average of the time-series standard deviation of the residuals in a 12-month rolling window regression of returns on the value-weighted HF return. We report the equal-weighted excess return. Source: Thomson-Reuters Lipper Hedge Fund Database.

	Pre-crisis		Crisis		Post-Crisis
	<i>Average</i>	<i>Start</i>	<i>Average</i>	<i>End</i>	<i>Average</i>
Panel A: Excluding Multi-Asset Funds					
XS Vol	8.52	5.66	16.15	20.14	7.29
TS Vol	5.20	3.76	7.19	11.21	4.58
Mean EW Returns	9.88	9.41	-6.17	-12.79	7.29
Panel B: Excluding Multi-Asset and Funds without U.S. Focus					
XS Vol	7.83	4.21	13.17	15.51	6.46
TS Vol	4.86	2.81	6.70	9.95	4.37
Mean EW Returns	7.34	6.98	0.59	-4.36	6.00
Panel C: Including All MBS Funds					
XS Vol	7.91	4.90	13.70	16.95	7.09
TS Vol	4.54	3.79	6.29	9.33	4.22
Mean EW Returns	6.23	7.15	-10.18	-18.23	3.22

some industry statistics. Prior to the crisis, there were 159 single-strategy MBS hedge funds. This number declined to 147 at the end of the crisis. In logs, this represents a 8 points decline. The mean AUM declines from 361 \$ million to 204 from start to end. In logs, this represents a 57 point decline. Again, results are similar excluding funds without a US focus, or including multi-strategy funds.

We show that our model, when subjected to an unexpected shock to volatility, can generate a loss of wealth, an increase in alpha and Sharpe ratios, and a decline in the participation rate. The decline in the participation rate following a shock to volatility is unique to our model, relative to models with exogenous segmentation, or fixed heterogeneity in participation costs or risk aversion. Similar to models of firm dynamics with entry costs, the greater risk bearing capacity of expert investors acts as an entry-deterrent. If expertise is complementary to complexity, meaning that more sophisticated investors are better able to hedge increases in risk, then less sophisticated investors are forced to exit when volatility increases. It is crucial to note that we are not emphasizing that excess returns or Sharpe ratios will be higher if participation is limited. The puzzle is the converse: Why is participation low in the cross section in asset classes with attractive risk return tradeoffs, and in the time series when this tradeoff becomes attractive?

We define a complex asset as one that imposes a significant amount of non-diversifiable but idiosyncratic risk on risk-averse investors. More complex assets impose more asset-specific risk on investors holding the asset. Merton (1987) was first to point out that idiosyncratic risk will be priced when there are costs associated with learning about or hedging a specific asset. There is a growing empirical literature that documents the importance of idiosyncratic risk in complex asset strategies (see, e.g., Titman and Tiu, 2011; Lee and Kim, 2014, for evidence of the importance of idiosyncratic risk in hedge fund returns). Pontiff (2006) investigates the role of idiosyncratic risk faced by arbitrageurs in a review of the literature and argues that “The literature demonstrates that idiosyncratic risk is the single largest cost faced by arbitrageurs”. Greenwood (2011) states that “Arbitrageurs are specialized and must be compensated for idiosyncratic risk,” and lists this first as the key friction investors in complex strategies face. According to Gabaix, Krishnamurthy, and Vigneron (2007), MBS returns are driven in large part by limits to arbitrage. Pontiff (1996, 2006); Greenwood (2011) cites evidence of idiosyncratic risk as a limit to arbitrage. To paraphrase Emanuel Derman, if you are using a model, you are short volatility, since you will lose money when your model is wrong (Derman (2016)). Certainly, the evidence in Table 1 shows that MBS hedge fund investors face considerable idiosyncratic risk.

Exposure to idiosyncratic risk arises because investing in complex assets requires a model.

Table 2: MBS Hedge Funds. Sample: Monthly data from 1991.12-2018.08. Includes all funds that report an investment focus on MBS. The crisis is defined as the period from 2007.6-2009.6. We report the averages for the pre-crisis, crisis, and post-crisis periods, as well as the 12-month average prior to the start and the end of the crisis. Source: Thomson-Reuters Lipper Hedge Fund Database.

	Pre-crisis		Crisis		Post-Crisis
	<i>Average</i>	<i>Start</i>	<i>Average</i>	<i>End</i>	<i>Average</i>
Panel A: Excluding Multi-Asset Funds					
Mean AUM (in logs)	19.04	19.71	19.39	19.14	19.60
Std AUM (in logs)	20.06	20.67	20.19	19.76	20.06
Number (in logs)	4.47	5.07	5.02	4.99	4.50
Mean AUM (in \$ millions)	185.85	361.24	264.58	204.78	324.29
Std AUM (in \$ millions)	514.21	951.61	585.80	380.54	513.30
Number	86.99	159.88	151.80	147.19	90.04
Panel B: Excluding Multi-Asset and Funds without U.S. Focus					
Mean AUM (in logs)	18.92	19.41	19.17	19.05	19.75
Std AUM (in logs)	19.45	19.91	19.57	19.44	20.12
Number (in logs)	4.05	4.39	4.29	4.26	3.89
Mean AUM (in \$ millions)	165.21	270.11	211.23	187.23	376.43
Std AUM (in \$ millions)	280.91	441.59	314.88	277.16	546.15
Number	57.25	80.37	73.06	71.06	49.13
Panel C: Including All MBS Funds					
Mean AUM (in logs)	19.22	19.64	19.31	19.18	19.56
Std AUM (in logs)	20.48	20.83	20.28	20.14	20.13
Number (in logs)	5.26	6.16	6.08	6.01	5.12
Mean AUM (in \$ millions)	223.37	340.03	242.29	212.89	312.00
Std AUM (in \$ millions)	787.94	1116.50	638.95	557.96	553.33
Number	191.69	471.51	436.97	407.05	168.11

We derive a specific source for the idiosyncratic risk in complex assets based on the returns earned from long position in a fundamental asset with some mispricing, or aggregate risk-adjusted “alpha”, and a short position in an imperfect tracking portfolio. We develop an industry equilibrium model of complex asset markets which generates higher alpha, higher Sharpe ratios, and lower participation for more complex assets in the cross section, and an increase in alphas and Sharpe ratios accompanied by a decline in participation following an unexpected shock to volatility. The model’s key features are investors with heterogeneous expertise, and free entry. Our model economy is populated by a continuum of risk-averse agents who choose to be either non-experts who can invest only in the risk free asset, or experts who can invest in both the risk free and risky assets. Experts differ ex ante in their level of expertise. On average, all expert investors in the complex asset earn the common market-clearing equilibrium return, but their returns are subject to investor-specific (or strategy-specific) shocks. Expertise shrinks the investor-specific volatility of the complex asset return, and, as a result, more expert investors earn a higher Sharpe ratio. The interpretation we propose is that expert investors build better hedging or tracking portfolios, with lower residual risk.²

In our model, all risk is investor-specific and idiosyncratic risk is priced. Funds cannot be reallocated across individual risk-averse investors. Since the risk in our economy is idiosyncratic, pooling this risk would eliminate the risk premium that experts require to hold it. For incentive reasons, asset managers cannot hedge their own exposure to their particular portfolio.³ This motivates why we endow expert investors in our model with CRRA preferences, but we do not model the principal-agent relation between the outside investors and asset managers.

Expert demand and risk bearing capacity act as a barrier to entry below a threshold level of expertise. Market clearing returns must compensate participating investors for the investor-specific risk they face, but less expert investors may not be adequately compensated, because equilibrium wealth-weighted demand from higher expertise investors depresses required returns. There is free entry, but, even in the long run, participation is limited and alpha is not dissipated. We characterize the equilibrium mapping from the endogenous joint distribution of expertise and financial wealth to complex asset prices. Importantly, in our dynamic equilibrium model, as opposed to in any static model, the stationary distribution for wealth conditional on expertise is endogenous. This is especially important when studying transition dynamics. Moreover, our tractable industry equilibrium model allows for us to characterize this distribution in closed

²Consistent with our model of complex asset returns, Pontiff (1996) finds that closed end mutual funds with portfolios that are more difficult to replicate have values that deviate more from fundamentals.

³In fact, Panageas and Westerfield (2009) and Drechsler (2014) provide important results for the portfolio choice of hedge fund managers who earn fees based on assets under management and portfolio performance. In particular, they show that these managers behave like constant relative risk aversion investors. These results extend the analysis of the impact of high-water marks in Goetzmann, Ingersoll, and Ross (2003).

form, and to perform comparative statics that take equilibrium changes in the distribution of wealth into account.

More complex assets impose more idiosyncratic risk on investors because they are harder to model. As a result, more complex assets earn higher equilibrium returns. The fact that assets with higher idiosyncratic risk require higher returns is unsurprising. Our model also has predictions about the equilibrium compensation *per unit of risk*, or Sharpe ratio. In theory, excess returns relative to risk may either increase or decrease as complexity (and hence risk) increases. Indeed, we show that market-level Sharpe ratios, which aggregate the individual Sharpe ratios of market participants, increase with asset complexity only if expertise and complexity are complementary, meaning that a marginal increase in idiosyncratic risk increases the rents from expertise. To clear the market at the higher risk level, excess returns must increase. However, if expertise and complexity are complementary, as complexity and therefore risk increase, participation declines as inexpert investors are driven out. The selection effect—the exclusion of lower expertise investors—attenuates the negative effect of increased risk on the market-level Sharpe ratio and ensures that this ratio increases. On the other hand, if expertise and complexity are not complementary, an increase in risk leads to *increased* participation, and both the direct effect of higher risk, and the selection effect of inexpert investors, drive Sharpe ratios lower. Thus, we argue that persistently elevated Sharpe ratios should only appear in markets for which expertise is more valuable as complexity increases.

We trace out the impulse response function to unanticipated volatility shocks by solving for the transition between steady-states in a calibrated version of the model. We show that a transitory increase in idiosyncratic volatility significantly reduces participation and increases the Sharpe ratio, but only when complexity and expertise are complements. In this case, Sharpe ratios only increase for high expertise agents; the increase in alpha is common to all investors, but effective volatility increases more for less sophisticated investors. As a result, a volatility shock induces those incumbent investors that are close to the participation margin to exit, in spite of the increase in alpha. As the remaining incumbents collect higher alpha, they rebuild their wealth and the risk-bearing capacity in the market, pushing the Sharpe ratio back down. The subsequent decrease in volatility triggers entry on the part of new experts, and the economy returns to its original steady-state.

The selection effect induced by endogenous exit of marginal experts is critical to our results. When we shut down the extensive participation margin, the model counterfactually predicts an initial decline in the Sharpe ratio, because the increase in effective volatility outweighs the increase in alpha. To the best of our knowledge, our paper is the first to highlight how this selection mechanism produces an increase in the price of idiosyncratic risk in response to an

increase the quantity of that risk, given that there exists complementarity. We also note that the role of idiosyncratic risk, and heterogeneity in performance, in driving asset dynamics during the financial crisis appears to be empirically relevant but has not previously been emphasized.

2 Related Literature

Our paper contributes to a large and growing literature on segmented markets and asset pricing. Relative to the existing literature, we provide a model with endogenous entry, a continuous distribution of heterogeneous expertise, and a rich distribution of expert wealth that is determined in stationary equilibrium. Thus, we have segmented markets, but allow for a participation choice. Our market has limited risk bearing capacity, determined in part by expert wealth, but in addition to the amount of wealth, the efficiency of the wealth distribution also matters for asset pricing. While substantial recent attention has been paid to slow moving capital, and to temporarily elevated basis spreads, we argue that it is equally important to understand “permanent alpha” and markets which remain segmented in the long run. Our model can explain both persistent market dislocations, and permanently elevated Sharpe ratios in complex asset markets.

We group the existing literature studying segmented markets and asset pricing into three main categories, namely investor heterogeneity, financial constraints and limits to arbitrage, and segmented market models with alternative micro-foundations to agency theory. There is also important and closely related work on heterogeneity in trading technologies and risk aversion (see, e.g., Dumas, 1989; Basak and Cuoco, 1998; Kogan and Uppal, 2001; Chien, Cole, and Lustig, 2011, 2012).

Our study shares the goal of understanding the returns to complex assets and strategies, and the features of segmented markets and trading frictions, with the literature on limits of arbitrage. Gromb and Vayanos (2010a) provide a recent survey of the theoretical literature on limits to arbitrage, starting with the early work by Brennan and Schwartz (1990) and Shleifer and Vishny (1997) (see also Aiyagari and Gertler, 1999; Froot and O’Connell, 1999; Basak and Croitoru, 2000; Xiong, 2001; Gromb and Vayanos, 2002; Yuan, 2005; Gabaix, Krishnamurthy, and Vigneron, 2007; Mitchell, Pedersen, and Pulvino, 2007; Acharya, Shin, and Yorulmazer, 2009; Kondor, 2009; Duffie, 2010; Gromb and Vayanos, 2010b; Hombert and Thesmar, 2014; Mitchell and Pulvino, 2012; Pasquariello, 2014; Kondor and Vayanos, 2014). Shleifer and Vishny (1997) emphasize that arbitrage is conducted by a fraction of investors with specialized knowledge, but similar to He and Krishnamurthy (2012), they focus on the effects of the agency frictions between arbitrageurs and their capital providers.

Although we do not explicitly model risks to the liability side of investors' balance sheets, one can interpret the shocks agents in our model face to include idiosyncratic redemptions. The asset pricing impact of financially constrained intermediaries has been studied in the literature on intermediary asset pricing following He and Krishnamurthy (2012, 2013) (see also, for example, Adrian and Boyarchenko, 2013). For empirical applications, see for example, Adrian, Etula, and Muir (2014), Muir (2014), and He, Kelly, and Manela (2017).

Finally, several papers develop alternative micro-foundations to agency theory for segmented markets. Allen and Gale (2005) provides an overview of their theory of asset pricing based on "cash-in-the-market". Plantin (2009) develops a model of learning by holding. Duffie and Strulovici (2012) develop a theory of capital mobility and asset pricing using search foundations. Glode, Green, and Lowery (2012) study asset price dynamics in a model of financial expertise as an arms race in the presence of adverse selection. Kurlat (2016) studies an economy with adverse selection in which buyers vary in their ability to evaluate the quality of assets on the market, and, like us, emphasizes the distribution of expertise on the equilibrium price of the asset. Gârleanu, Panageas, and Yu (2015) derive market segmentation endogenously from differences in participation costs. Edmond and Weill (2012), Haddad (2014), and DiTella (2016) study the effects of idiosyncratic risk from concentrated holdings on asset prices. Kacperczyk, Nosal, and Stevens (2014) construct a model of consumer wealth inequality from differences in investor sophistication. We are the first in this literature to study the effect of selection among expert investors in response to an increase in idiosyncratic risk, which appears to be empirically relevant. We also show in the online appendix that comparing our model with heterogeneity in effective volatilities to models with heterogeneous risk aversion or participation costs, only our model can jointly explain the observed patterns of (1) exit following a shock to volatility (2) heterogeneity in portfolio choice (equivalently, in leverage ratios) and (3) heterogeneity in fund performance.

In terms of methodology, our model is an example of an "industry equilibrium" model in the spirit of Hopenhayn (1992a,b). Such models are typically used to study the role of firm dynamics, entry, and exit in determining equilibrium prices in an environment which builds on the heterogeneous agent framework developed in Bewley (1986). This literature focuses in large part on explaining firm growth, and moments describing the firm size distribution. Recent progress in the firm dynamics literature using continuous time techniques to solve for policy functions and stationary distributions include Miao (2005); Luttmer (2007); Gourio and Roys (2014); Moll (2014); Achdou, Han, Lasry, Lions, and Moll (2014). We draw on results in these papers as well as discrete time models of firm dynamics, as in recent work by Clementi and Palazzo (2016), which emphasizes the role of selection in explaining the observed relationships

between firm age, size, and productivity. We are the first to use a model in this class to study the size or wealth distribution of financial intermediaries.

Recently, dynamic heterogeneous agent models have also been used to study wealth distributions in the consumer sector. Although our focus is on the application to the long-run equilibrium pricing of complex assets, we make several theoretical contributions to this literature. We build on work by Benhabib, Bisin, and Zhu (2011, 2015, 2016),⁴ which study the wealth distribution of consumers subject to idiosyncratic shocks to capital and/or labor income risk. These papers contain fundamental results describing necessary conditions for fat-tailed or Pareto wealth distributions, as well as interesting positive results about the importance of capital income shocks and taxation schemes in shaping observed wealth inequality. (See Gabaix, 2009, for a review of the empirical evidence and theoretical foundations for power law distributions in economics and finance.)

A key distinction of our work is that we clear the market for the risky asset, and solve the resulting fixed point problem determining equilibrium excess returns and market volatility, consistent with our interpretation of the complex risky asset market we study. Another distinction is that we study a participation decision, which plays a crucial role in our model's equilibrium price and allocation outcomes. Our paper also introduces heterogeneity in technologies, which leads to endogenous variation in the drift and volatility of wealth across agents. Gabaix, Lasry, Lions, and Moll (2016) also emphasize the importance of heterogeneity in income growth rates in generating realistic inequality. They do not feature a portfolio choice, or endogenous variation in growth rates, and thus our study complements theirs. See also Toda (2014) and Toda and Walsh (2015) for related results describing consumer wealth distributions using double power laws. Finally, another distinction is that we draw on the techniques used in Gabaix (1999), who studies city size distributions in a model in which relative sizes follow a reflecting, or regulated, geometric Brownian motion. Because the stationary distribution in our model depends on the equilibrium fixed point excess return on the risky asset, it is more convenient for us to employ a reflecting barrier for relative wealth levels, rather than Poisson elimination as in the Benhabib, Bisin, Zhu (and many other) papers.

Finally, our paper also makes contact with a mature literature on survival of traders with heterogeneous beliefs (see, e.g., Sandroni, 2000; Kogan, Ross, Wang, and Westerfield, 2006; Borovicka, 2019). In our model, investors have access to heterogeneous hedging technologies. Less expert investors are gradually driven out of the market by more expert investors who face less volatile returns, invest more aggressively and accumulate wealth faster. We neutralize this force by imposing a lower bound on investor wealth. This bound allows us to derive a

⁴See also Kaplan, Moll, and Violante (2016) for a model with a portfolio choice over liquid and illiquid assets.

stationary equilibrium, but has only a minor impact on the pricing of risk, because investors at the lower bound are negligibly small. In the absence of this bound, the most expert investor would accumulate all of the wealth in the limit.

3 Model of a Complex Asset Market

According to its general definition, true α cannot be generated by bearing systematic risk. However, capturing α is risky. We provide a specific source for the idiosyncratic risk in complex assets by modeling investment in complex assets as long-short strategies which attempt to “harvest alpha” in a fundamental asset by going long the fundamental asset, and short a tracking portfolio that eliminates systematic or aggregate risk, but is fairly priced.

More generally, it can be argued that complex assets expose their owners to idiosyncratic risk through several channels. First, any investment in a complex asset requires a joint investment in the front and back office infrastructure necessary to implement the strategy. Second, their constituents tend to be significantly heterogeneous, so that no two investors hold exactly the same asset. Third, the risk management of complex assets typically requires a hedging strategy that will be subject to the individual technological constraints of the investor. Hedging portfolios, to cite just one example, tend to vary substantially across different investors in the same asset class. Fourth, firms which manage complex assets may be exposed to key person risk due to the importance of specialized traders, risk managers, and marketers. Broadly interpreted, these risks may come either from the asset side, or from the liability side, since funding stability likely varies with expertise. However, we abstract from the micro-foundations of risks from the liability side of funds’ balance sheets, and model risk on the asset side.

3.1 Preferences, Endowments, & Technologies

We study a model with a continuum of investors of measure one, with CRRA utility functions over consumption:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Panageas and Westerfield (2009) and Drechsler (2014) analyze the portfolio choice of hedge fund managers who earn fees based on assets under management and portfolio performance. In particular, they show that these managers behave like constant relative risk aversion investors. These results extend the analysis of the impact of high-water marks in Goetzmann, Ingersoll, and Ross (2003).

Investment Technology Investors are endowed with a level of expertise which varies in the cross section, but is fixed for each agent over time. Each individual investor is born with a fixed expertise level, x , drawn from a distribution with pdf $\lambda(x)$, and a continuous cdf $\Lambda(x)$. Investors can choose to be experts, and have access to the complex risky asset, or non-experts, who can only invest in the risk free asset.

For convenience, we also assume that the support of expertise is bounded above by \bar{x} , although most of our results only require that $\sigma(x)$ satisfies $\lim_{x \rightarrow \infty} \sigma(x) = \underline{\sigma} > 0$. The implied lower bound on volatility, $\underline{\sigma}$, represents complex asset risk that cannot be eliminated even by the agents with the greatest expertise, and it guarantees that the growth rate of wealth is finite.

Assumption 1 *The support of expertise is bounded above by \bar{x} .*

Each investor’s complex risky asset delivers a stochastic return which follows a geometric Brownian motion:

$$dR_i(t) = [r_f + \alpha(t)] dt + \sigma(\sigma_\nu, x) dB_i(t) \quad (1)$$

where $\alpha(t)$ is the common excess return on the risky asset at time s with initial time t and $B_i(t)$ is a standard Brownian motion which is investor-specific and i.i.d. in the cross section. Hereafter, we suppress the dependence of the Brownian shock on investor i in our notation for parsimony. The effective volatility of the risky technology, $\sigma(\sigma_\nu, x)$ is a function of the volatility of the complex asset before expertise is applied, and the specific investor’s level of expertise, x . We use the notation σ_ν and the language “total volatility” to describe the overall level of volatility of a complex asset class, before expertise is applied. “Effective volatility” is, then, a function of the asset’s “total” volatility, and the investor’s level of expertise. For now, we focus on describing the equilibrium for a single asset, and we suppress the positive dependence of $\sigma(x)$ on the total volatility of the asset class σ_ν . Below, we describe comparative statics across assets with varying complexity, with more complex assets characterized by a higher σ_ν , or “total volatility”. We refer to $\sigma(x)$ as “effective volatility”, meaning the remaining volatility the investor faces after expertise has been applied. In particular, we assume that the volatility of the risky technology $\sigma(x)$ decreases in the investor’s level of expertise x , i.e. $\frac{\partial \sigma(x)}{\partial x} < 0$.

A simple motivation for the return process in Equation (1) is that in order to invest in the risky asset and to earn the common market clearing return, an investor must also jointly invest in a technology with a zero mean return and an idiosyncratic shock. This technology represents each investor’s specific hedging and financing technologies, as well as the unique features of their particular asset. We also provide a specific formal micro-foundation for Equation (1) in Lemma 1. Investors take a long position in an underlying asset with some alpha or mis-pricing relative to its systematic risk exposure, and a short position in an imperfectly correlated, investor-

specific, tracking portfolio. The long-short position is designed to “harvest the alpha” in the underlying asset while hedging out unnecessary aggregate risk exposures to the agent’s best ability. It can also be interpreted as a relative value strategy aimed at over and under weighting different parts of the cross section in order to capture misvaluation while attempting to hedge out market level risk, as is common practice in quantitative equity and fixed income strategies. Consistent with our motivation, Fung and Hsieh (1997) report that hedge fund returns, unlike mutual fund returns, have a low, and indeed sometimes negative, correlation with asset class returns.

Investors with more expertise have superior tracking portfolios which hedge out more of the risk in the underlying asset. As a result, their total net position is less risky and they earn the common market clearing alpha while bearing less risk. An additional contribution of our paper is to provide a precise explanation for the idiosyncratic risk that the prior literature has argued is important for understanding complex asset returns.

Derivation 1 *Each investor’s investor-specific return process, given by (1), can be derived as the net return on a strategy in which the investor takes a long position in an underlying fundamental asset, and a short position in a tracking portfolio. The returns to the fundamental asset are given by:*

$$\frac{dF(t)}{F(t)} = [r_f + \alpha(t) + a(t)] dt + \sigma^F dB^F(t). \quad (2)$$

The returns to each investor’s best per-unit tracking portfolio is given by:

$$\frac{dT_i(t)}{T_i(t)} = a(t)dt + \rho_i(x)\sigma^F dB^F(t) - \sigma^T(x)dB_i^T(t), \quad (3)$$

where $dB^F(t)$ and $dB_i^T(t)$ are independent Brownian motions, and $\rho_i(x)$ represents the dependence of the investor’s tracking portfolio returns on the fundamental risk in the underlying asset. The correlation between the investor’s tracking portfolio return and underlying asset return is given by

$$\text{corr} \left(\frac{dF}{F} - \frac{dT_i}{T_i}, \frac{dF}{F} \right) = \frac{(1 - \rho_i(x))\sigma^F}{\sqrt{((1 - \rho_i(x))\sigma^F)^2 + (\sigma^T(x))^2}},$$

which is increasing in $\rho_i(x)$. We assume that $\frac{\partial|\rho_i(x)-1|}{\partial x} < 0$, so that higher expertise investors have access to tracking portfolios which hedge out more fundamental risk. The net asset returns are given by:

$$dR_i(t) = \frac{dF(t)}{F(t)} - \frac{dT_i(t)}{T_i(t)} = [r_f + \alpha(t)] dt + \sigma(x)dB_i(t), \quad (4)$$

where

$$\sigma(x)dB_i(t) \equiv (1 - \rho_i(x))\sigma^F dB^F(t) + \sigma^T(x)dB_i^T(t). \quad (5)$$

The return process in Equations (4)-(5) generates a single process for expertise level wealth dynamics in which aggregate risk washes out, and all investors with expertise level x face the same amount of effective risk, under the sufficient condition in Assumption 2:

Assumption 2 At each level of expertise, half of investors over-hedge (type o with $\rho_o(x) > 1$), and half under-hedge (type u with $\rho_u(x) < 1$). That is,

$$\frac{\rho_o(x) + \rho_u(x)}{2} = 1.$$

Lemma A.1, which we write formally and prove in the Appendix, states that, given the derivation of the return process in Equation (1) provided in Lemma 1, aggregate shocks do not affect equilibrium policy functions or prices, and therefore the return dynamics Equation (1) are taken as given in the remainder of the main text.⁵

Note that, by definition, if $\rho_i(x) < 1$, i.e. the tracking portfolio returns are not perfectly correlated with the underlying asset returns (in which case there would exist a riskless arbitrage opportunity), then the tracking portfolio introduces risk which is independent from fundamental risk. We assume this independent risk is uncorrelated across investors. Because each investor has their own model and strategy implementation, tracking portfolios introduce investor-specific shocks. Formally, we use the fact that any Brownian shock which is partially correlated with the underlying fundamental Brownian shock $dB^F(t)$ can be decomposed into a linear combination of a correlated shock and an independent shock. We denote this independent, investor-specific shock $dB_i^*(t)$. The amount of idiosyncratic risk the tracking portfolio introduces is smaller the closer to one is $\rho_i(x)$, which is intuitive. We assume that agents with higher expertise have better tracking portfolios which eliminate more risk from the underlying fundamental shock. Across asset classes, more complex assets are characterized by more imperfect models and tracking portfolios, and hence highly complex assets impose more total risk σ_ν on investors. This higher total risk can result from overall lower quality tracking portfolios (lower $\rho_i(x)$). Higher total risk can also result from the greater model and execution risk of highly complex assets, which can be considered to be part of $\sigma^T(x)$.

To be an expert, an investor must pay the entry cost F_{nx} to set up their specific technology for investing in the complex risky asset. Experts must also pay a maintenance cost,

⁵We can allow for another source of aggregate risk without changing the equilibrium wealth dynamics provided that more expert investors do not have an advantage in bearing this other type of aggregate risk (see, e.g., Krueger and Lustig (2009)).

F_{xx} to maintain continued access to the risky technology. We specify that both the entry and maintenance costs are proportional to wealth:

$$\begin{aligned} F_{nx} &= f_{nx}w, \\ F_{xx} &= f_{xx}w, \end{aligned}$$

which yields value functions which are homogeneous in wealth.

Optimization We first describe the Bellman equations for non-experts and experts respectively, and characterize their value functions, as well as the associated optimal policy functions. With the value functions of experts and non-experts in hand, we then characterize the entry decision.

We begin with non-experts, who can only invest in the risk free asset. Let $w(t)$ denote the wealth of investors at time t with initial wealth W_0 at time 0. The riskless asset delivers a fixed return of r_f . All investors choose consumption, and an optimal stopping, or entry time according to the Bellman Equation:

$$V^n(w(t), x) = \max_{c^n(t), \tau} \mathbb{E} \left[\int_t^\tau e^{-\rho(s-t)} u(c^n(s)) ds + e^{-\rho(\tau-t)} V^x(w(t, \tau) - F_{nx}, x) \right] \quad (6)$$

$$\text{s.t. } dw(t) = (r_f w(t) - c^n(t)) dt \quad (7)$$

where ρ is their subjective discount factor, $c^n(t)$ is consumption at time t , F_{nx} is the entry cost, and τ is the optimal entry date. Under the assumptions of linear entry and maintenance costs, and expertise which is fixed over time, the optimal entry date in a stationary equilibrium will be either immediately, or never. Thus, assuming an initial stationary equilibrium, investors who choose an infinite stopping time are then non-experts, and investors who choose a stopping time $\tau = t$ are experts.

Experts allocate their wealth between current consumption, a risky asset, and a riskless asset. They also choose an optimal stopping time T to exit the market.

$$V^x(w(t), x) = \max_{c^x(x,t), T, \theta(x,t)} \mathbb{E} \left[\int_t^T e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(T-t)} V^n(w(t), x) \right] \quad (8)$$

$$\begin{aligned} \text{s.t. } dw(t) &= [w(t)(r_f + \theta(x, t)\alpha(t)) - c^x(x, t) - F_{xx}] dt \\ &+ w(t)\theta(x, t)\sigma(x) dB(t), \end{aligned} \quad (9)$$

where $\alpha(t)$ is the equilibrium excess return on the risky asset, $\theta(x, t)$ is the portfolio allocation to the risky asset by investors with expertise level x , $c^x(x, t)$ is consumption, F_{xx} is the main-

tenance cost. Neither entry nor exit will occur in a stationary equilibrium. However, we show that exit, followed later by re-entry, occurs following an unexpected transitory shock to total volatility in Section 5.2 where we study transition dynamics.

The following proposition states the analytical solutions for the value and policy functions in our model. We prove this Proposition by guess and verify in the Appendix.

Proposition 1 *Value and Policy Functions:* *The value functions are given by*

$$V^x(w(t), x) = y^x(x, t) \frac{w(t)^{1-\gamma}}{1-\gamma} \quad (10)$$

$$V^n(w(t), x) = y^n(x, t) \frac{w(t)^{1-\gamma}}{1-\gamma} \quad (11)$$

where $y^x(x)$ and $y^n(x)$ are given by:

$$y^x(x) = \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)} \right]^{-\gamma} \text{ and} \quad (12)$$

$$y^n(x) = \left[\frac{(\gamma-1)r_f + \rho}{\gamma} \right]^{-\gamma}. \quad (13)$$

The optimal policy functions $c^x(x, t)$, $c^n(t)$, and $\theta(x)$ are given by:

$$c^x(x, t) = [y^x(x)]^{-\frac{1}{\gamma}} w(t), \quad (14)$$

$$c^n(t) = [y^n(x)]^{-\frac{1}{\gamma}} w(t) \text{ and} \quad (15)$$

$$\theta(x, t) = \frac{\alpha(t)}{\gamma\sigma^2(x)}. \quad (16)$$

Furthermore, the wealth of experts evolves according to the law of motion:

$$\frac{dw(t)}{w(t)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t)}{2\gamma^2\sigma^2(x)} \right) dt + \frac{\alpha(t)}{\gamma\sigma(x)} dB(t) \quad (17)$$

Finally, investors choose to be experts if the excess return earned per unit of wealth exceeds the maintenance cost per unit of wealth:

$$\frac{\alpha^2(t)}{2\sigma^2(x)\gamma} \geq f_{xx}. \quad (18)$$

We define the lowest level of expertise amongst participating investors to be \underline{x} , the level of expertise for which Equation (18) holds with equality. Note that the law of motion for wealth is a sort of weighted average of the return to the risky and riskless assets, as determined by

portfolio choice, net of consumption. The drift and volatility of investors' wealth are increasing in the allocation to the risky asset.

In our model, wealth dynamics are driven by portfolio performance. In practice, asset manager wealth is also affected by fund inflows and outflows. Although one can interpret effective volatility to include shocks from fund flows, these shocks are not performance related. Introducing a feedback effect between performance and wealth breaks the linearity of the return process which enables analytic characterizations. In the Online Appendix, we present a specification of the model with exogenous fund flows that are increasing and concave in expertise. For this specification, our main results are preserved. We also note that empirical results on fund flows and performance within the hedge fund space actually appear to be consistent with our main specification. In particular, Fung, Hsieh, Naik, and Ramadorai (2008) report that, while hedge fund fund-of-funds that load on systematic risk exhibit a statistically significant relationship between fund flows performance, flows into funds which *have alpha* show *no evidence* of return-chasing behavior, i.e. in their study the coefficient of quarterly flows on lagged quarterly returns is not statistically significant. They argue that this difference is due to the greater sophistication of institutional hedge fund investors, relative to retail mutual fund investors.

3.2 The Distribution(s) of Expert Wealth

The total amount of wealth allocated to the complex risky asset, as well as the distribution of expert wealth across expertise levels, are key aggregate state variables for the the first and second moments of the equilibrium returns to the complex risky asset. Once the participation decision has been made, given that we do not clear the market for the riskless asset, the wealth of non-experts is irrelevant for the returns to the complex risky asset. We solve for the cross sectional distribution of expert wealth in a stationary equilibrium of our model. Given that expertise is fixed over time for each investor, constructing the wealth distribution at each expertise level is sufficient to obtain the cross-sectional joint distribution of wealth and expertise.

Given that aggregate wealth on average grows over time, we will study the stationary equilibrium of a detrended economy in which the distribution of normalized wealth is stationary. Define normalized wealth $z(t)$ as:

$$z(t) \equiv \frac{w(t)}{\exp(g_S t)}, \quad (19)$$

where g_S is the exogenous growth rate of an exogenously given stock variable, which we normalize to one. We will discuss these parameters in more detail when we turn to the description of the stationary equilibrium. We can then replace wealth w with normalized wealth z in the value

and policy functions without consequence, due to the homogeneity described in Proposition 1. The ratio $z(t)$ follows a geometric Brownian motion given by

$$\frac{dz(t)}{z(t)} = \frac{dw(t)}{w(t)} - g_S dt = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1) \alpha^2(t)}{2\gamma^2 \sigma^2(x)} - g_S \right) dt + \frac{\alpha(t)}{\gamma \sigma(x)} dB(t). \quad (20)$$

Let the cross-sectional p.d.f. of expert investors' wealth and expertise at time t be denoted by $\phi^x(z, x, t)$. Without additional assumptions, the normalized wealth of lower expertise agents will shrink to zero. Two methods are commonly used to generate a stationary distribution by effectively replacing agents whose wealth grows too small. The first, for example used in Benhabib, Bisin, and Zhu (2016), is to employ a life cycle model, or Poisson elimination of agents. The second, employed by Gabaix (1999), is to introduce a reflective or regulating barrier at a minimum relative size. We follow the strategy for constructing a stationary equilibrium in that paper, and impose a reflective, or regulating, barrier at a minimum normalized wealth, z_{\min} .

Constructing a stationary distribution for normalized wealth requires an assumption that the growth rate in the denominator in (19) is sufficiently high, ensuring that z is bounded for all agents.⁶ We begin by defining the law of motion for the growth rate of mean wealth of agents with a given level of expertise x , which is given by

$$\frac{d\mathbb{E}[w|x(t)]}{\mathbb{E}[w|x(t)]} \equiv [g(x)] dt.$$

where $g(x)$ is determined in equilibrium. Then, denote the average growth rate amongst agents with the highest level of expertise as $g(\bar{x})$. We assume that:

Assumption 3 $g_S > g(\bar{x})$.

This assumption, which we verify holds in our calibrated exercises, ensures that the drift of normalized wealth in Equation (20) is negative for every agent. Given Assumption 3, the growth rate of any individual agent's normalized wealth, even those with the highest level of expertise, will grow more slowly than the mean wealth of the highest expertise agents when their wealth exceeds z_{\min} , since average wealth includes the effect of the reflecting barrier. Because each agent's normalized wealth grows at a negative rate, the reflecting barrier both prevents all experts from becoming infinitely small, while also ensuring that total wealth is finite.

We adopt the assumption of a minimum normalized wealth because it leads to a more elegant expression for the wealth distribution. In particular, normalized wealth conditional

⁶Gabaix (1999) constructs a model of the city size distribution in which total population grows at an exogenous rate. See also Section 1.10 of Harrison (2013) for the limit distribution of a reflected Brownian motion.

on expertise are characterized by a simple Pareto distribution. With Poisson elimination, the resulting stationary distribution can only be simplified to a mixture of two Pareto distributions. While such a distribution can be derived analytically (see Benhabib, Bisin, and Zhu (2011) and Benhabib, Bisin, and Zhu (2016)), it severely complicates the expressions used to analyze the model and provide comparative statics. We show in the Online Appendix that our quantitative results are insensitive to our calibration of z_{\min} around our calibrated value of 1%.

We assume that one of two things can happen when a fund reaches the minimum normalized size z_{\min} . Either the fund is liquidated, or it is rescued by an infusion of cash. In the Online Appendix, we show that the value at z_{\min} from adopting the optimal policy functions under Brownian normalized wealth dynamics is equal to the value of adopting those same policies given appropriately defined probabilities of liquidation vs. receiving new funds which enable the investor to remain in the market with a normalized wealth of z_{\min} . In the case of exit, we assume the investor becomes a non-expert and is replaced by a new entrant with normalized wealth z_{\min} and the same level of expertise x as the exiting agent. The perceived probability of rescue is increasing in wealth and expertise, as is intuitive.

Proposition 2 *The value and policy functions under reflecting, or regulated, Brownian dynamics for normalized wealth, given by:*

$$dz_t/z_t = \mu_z dt + \sigma_z dB_t \text{ for } z_t > z_{\min}$$

and

$$dz_t/z_t = \max(\mu_z dt + \sigma_z dB_t, 0) \text{ for } z_t \leq z_{\min},$$

with μ_z and σ_z defined in Equation (20), are given by the solution to the alternative model in:

$$\begin{aligned} V^r(z(t), x) &= \max_{c^r(x,t), T, \theta^r(x,t)} \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x, s)) ds \right. \\ &\quad \left. + e^{-\rho(s'-t)} [(1-q)V^r(z_{\min}, x) + qV^n(z(t), x)] \right] \\ \text{s.t. } dz(t) &= [z(t)(r_f + \theta^r(x, t)\alpha(t)) - c^r(x, t) - F_{xx}] dt \\ &\quad + z(t)\theta^r(x, t)\sigma(x)dB(t), \end{aligned}$$

where $s' < T$ is the first time that an agent's wealth falls below z_{\min} . The value and policy functions for this alternative model are equivalent to those under the true Brownian dynamics in the model of Equations (8) - (9), for the appropriately defined probability, q .

The proof, along with the definition of the punishment probabilities are given in the Online Appendix.

With this result in hand, we derive the stationary distributions for normalized wealth. The Kolmogorov forward equations describing the evolution of the wealth distributions, conditional on expertise level x and taking $\alpha(t)$ as given, can be stated as follows:⁷

$$\begin{aligned}
\partial_t \phi^x(z, x, t) &= -\partial_z \left[\left((r_f + \theta(x, t) \alpha(t)) - [y^x(x)]^{-\frac{1}{\gamma}} - f_{xx} - g_S \right) z \phi^x(z, x, t) \right] \\
&\quad + \frac{1}{2} \partial_{zz} \left([z \theta(x, t) \sigma(x)]^2 \phi^x(z, x, t) \right) \\
&= -\partial_z \left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1) \alpha^2(t)}{2\gamma^2 \sigma^2(x)} - g_S \right) z \phi^x(z, x, t) \right] \\
&\quad + \frac{1}{2} \partial_{zz} \left[\left(z \frac{\alpha(t)}{\gamma \sigma(x)} \right)^2 \phi^x(z, x, t) \right].
\end{aligned} \tag{21}$$

We then study the stationary distribution of normalized wealth, in which $\partial_t \phi^x(z, x, t) = 0 \forall x$. We take as given, for now, that $\alpha(t)$ will be constant, and we ensure that this is the case in the stationary equilibrium. A stationary distribution of normalized wealth $\phi^x(z, x)$ satisfies the following equation:

$$\begin{aligned}
0 &= -\partial_z \left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1) \alpha^2}{2\gamma^2 \sigma^2(x)} - g_S \right) z \phi^x(z, x) \right] \\
&\quad + \frac{1}{2} \partial_{zz} \left[\left(z \frac{\alpha}{\gamma \sigma(x)} \right)^2 \phi^x(z, x) \right].
\end{aligned} \tag{22}$$

We use guess and verify to show that the stationary distribution of wealth shares at each level of expertise is given by a Pareto distribution with an expertise-specific tail parameter. This tail parameter, which we denote by β , is determined by the drift and volatility of the expertise-specific law of motion for normalized wealth. Intuitively, the larger the drift and volatility of the expertise-specific wealth process, the fatter the tail of the wealth distribution at that level of expertise will be.

Proposition 3 *The stationary distribution of normalized wealth $\phi^x(z, x)$ has the following form:*

$$\phi(z, x) \propto C(x) z^{-\beta(x)-1},$$

⁷See Dixit and Pindyck (1994) for a heuristic derivation, or Karlin and Taylor (1981) for more detail.

where

$$\begin{aligned}\beta(x) &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\ C_1 &= 2\gamma(f_{xx} + \rho - r_f + \gamma g_S), \\ C(x) &= \frac{1}{\int z^{-\beta(x)} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{z_{\min}^{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma}}.\end{aligned}$$

See the Online Appendix for the Proof. Note that, in this proof, we also show that, in the stationary distribution, $\beta > 1$, which ensures a finite integral, and confirms that the distribution satisfies stationarity. The following Corollary solves for the tail parameter of the highest expertise agents, as well as the average growth rate amongst these agents. Given this growth rate, we can write the tail parameter for any level of expertise as a function of the coefficient of relative risk aversion, the minimum normalized wealth, and the ratio the effective variance relative to the effective variance for the highest expertise agents. This is convenient, because it clearly shows the dependence of the thickness of the right tail of the relative wealth distribution on expertise.

Corollary 1 *For the highest expertise agents, we have*

$$\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}} = C_1 \frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma$$

where \bar{z} is mean of normalized wealth of experts with highest expertise,

$$\bar{z} = \int_{z_{\min}}^{\infty} z \phi(z, \bar{x}) dz = z_{\min} \left[1 + \frac{1}{\beta(\bar{x}) - 1} \right]$$

For all other expertise levels, we have

$$\beta(x) = \left(\gamma + \frac{z_{\min}/\bar{z}}{1 - z_{\min}/\bar{z}} \right) \frac{\sigma^2(x)}{\sigma^2(\bar{x})} - \gamma.$$

The parameter $\beta(x)$ controls the rate of decay in the tail of each expertise-specific wealth distribution. We now have two expressions for this parameter:

$$\beta(x) = 2\gamma(f_{xx} + \rho - r_f + \gamma g_S) \frac{\sigma^2(x)}{\alpha^2} - \gamma, \tag{23}$$

from Proposition 3 and

$$\beta(x) = \left(\gamma + \frac{z_{\min}/\bar{z}}{1 - z_{\min}/\bar{z}} \right) \frac{\sigma^2(x)}{\sigma^2(\bar{x})} - \gamma. \quad (24)$$

from Corollary 1. The smaller is β , the more slowly the normalized wealth distribution decays, and the fatter is the upper tail. Both equations clearly show that $\beta(x)$ an increasing function of expertise level volatility, $\sigma(x)$, and thus a decreasing function of expertise, x . That is, the wealth distribution of experts with a higher level of fixed expertise has a fatter right tail. Investors with higher expertise allocate more wealth to the risky asset, which increases the mean and volatility of their wealth growth rate. Both a higher drift, and a wider distribution of shocks, lead to a fatter upper tail for wealth. Moreover, Equation (24), in which the dependence of the tail parameter on expertise is given by $\frac{\sigma^2(x)}{\sigma^2(\bar{x})}$, shows that if the relation between expertise and effective volatility is steeper, then the difference in the size of the right tails of the wealth distribution across expertise levels increases. In equilibrium, variation in effective volatilities in complex asset markets will be driven both by the functional form for effective volatility, and by participation decisions which determine how different effective volatilities of participating agents can be given equilibrium pricing.

3.3 Aggregation and Stationary Equilibrium

The equilibrium market clearing α is determined by equating supply and demand. In order to ensure that the supply of the complex risky asset does not become negligible as investor wealth grows, we assume that the supply grows at the same rate as the denominator of normalized wealth. This would be the case if the (unmodeled) supplier also grew with the size of the economy. Because we have constructed a stationary distribution for normalized wealth, in which the normalization variable grows at a rate of g_S (see Equation (19)), the asset supply must grow at this same rate in order to ensure that the market clears at a constant alpha.

Assumption 4 *The growth rate of risky asset supply is given by:*

$$\frac{dS(t)}{S(t)} = g_S dt,$$

where $g_S > g(\bar{x})$ as in Assumption 3.

We are now ready to define a stationary equilibrium for normalized, or detrended, economies. We define normalized aggregate investment in the complex risky asset to be I , defined as:

$$I \equiv \int \lambda(x) I(x) dx, \quad (25)$$

where $I(x)$ is the mean of detrended expertise-level investment in the complex risky asset, namely,

$$I(x) = \frac{\alpha}{\gamma\sigma^2(x)}Z(x), \quad (26)$$

and where $Z(x)$ is the mean of expertise-level normalized wealth,

$$Z(x) \equiv z_{\min} \left(1 + \frac{1}{\beta(x) - 1} \right).$$

This is the well-known expression for the mean of a Pareto distribution.⁸ The condition which determines the market clearing α in a stationary equilibrium of a detrended economy equates detrended investment to detrended asset supply.

Definition 1 *A stationary equilibrium consists of a market clearing α , policy functions for all investors, and a stationary distribution over investor types $i \in \{x, n\}$, expertise levels x , and normalized wealth z , $\phi(i, z, x, t)$, such that given an initial wealth distribution, an expertise distribution $\lambda(x)$, and parameters describing risk aversion, investors' subjective discount factor, the risk free rate, entry and maintenance costs, total volatility, minimum normalized wealth, a growth rate for the wealth normalization variable, and detrended risky asset supply, $\{\gamma, \rho, r_f, f_{nx}, f_{xx}, \sigma_\nu, z_{\min}, g_S$, and $S\}$, the economy satisfies:*

1. *Investor optimality: Investors choose participation in the complex risky asset market according to Equation (18), as well as optimal consumption and portfolio choices $\{c^n(t), c^x(x, t), \theta(x, t)\}_{t=0}^\infty$ according to Equations (14)-(16), such that their utilities are maximized.*
2. *Market clearing: In a stationary equilibrium, we have:*

$$S \equiv \frac{S(t)}{\exp(g_S t)} \quad \text{and} \quad I \equiv \int \lambda(x) I(x) dx = S, \quad (27)$$

3. *The distribution over all individual state variables is stationary, i.e. $\partial_t \phi(i, z, x, t) = 0$.*

4 Analytical Results

With policy functions, stationary distributions, and the equilibrium definition in hand, we turn to our asset pricing results.

⁸For a simple derivation, see the proof of Corollary 1.

4.1 Analytical Asset Pricing Results

We define a more complex asset as one that introduces more idiosyncratic risk. Comparing across assets, we use σ_ν to denote the total volatility of the asset before expertise is applied, so that the risk in each investor's asset is $\sigma(\sigma_\nu, x)$, and is increasing in the first (asset-specific) argument, and decreasing in the second (investor-specific) input. We provide specific examples below, but begin with any general function satisfying these two properties. Importantly, we describe conditions under which more complex assets, or assets which introduce more idiosyncratic risk, have lower participation despite higher α 's and higher Sharpe ratios. A key requirement is complementarity of expertise and complexity, meaning that the higher risk of more complex assets more negatively impacts investors with lower expertise.

We begin by studying comparative statics over the equilibrium market clearing α . Although we focus on comparative statics over total volatility, we also provide results for the market clearing α for changes other parameters which might proxy for asset complexity, such as the cost of maintaining expertise, or investor risk aversion. Next, we analyze individual Sharpe ratios. We emphasize across-investor heterogeneity in changes in the risk return tradeoff as total volatility changes. Because other parameters which might also vary with complexity, such as γ or f_{xx} , do not change investor-specific volatility, the results for individual Sharpe ratios are the same as those for α . Finally, we study market level Sharpe ratios, with a focus on the effects of changes in total volatility on the intensive and extensive margins of participation by investors with heterogeneous expertise.

Investor Demand, Aggregate Demand, and Equilibrium α We first describe the comparative statics for demand conditional on investors' expertise levels in Lemma 2.

Lemma 2 *Using Equation (26) for mean expertise-level investor demand conditional on expertise, x , we have following comparative statics, $\forall x$: $\frac{\partial I(x)}{\partial \sigma^2(x)} < 0$, $\frac{\partial I(x)}{\partial \sigma_\nu} < 0$, $\frac{\partial I(x)}{\partial \alpha} > 0$, $\frac{\partial I(x)}{\partial \gamma} < 0$, and $\frac{\partial I(x)}{\partial f_{xx}} < 0$.*

Demand for the risky asset at each level of expertise, which is simply $\lambda(x)I(x)$, is increasing in the squared investor-specific Sharpe ratio, and it is increasing in α . Demand is decreasing in effective variance, total volatility, risk aversion, and the maintenance cost.

With expertise-level total demands in hand, we can construct comparative statics for aggregate demand. We cannot express the equilibrium excess return in closed form. However, the following Proposition shows that the equilibrium excess return, α , and normalized aggregate demand, I , form a bijection. This uniqueness result in turn ensures that α can easily be solved for numerically as the unique fixed point to Equation (27).

Proposition 4 *Aggregate market demand for the complex risky asset is an increasing function of the excess return, α , and α and I form a bijection. Mathematically,*

$$\frac{\partial I}{\partial \alpha} > 0.$$

Proposition 5 provides comparative statics over the aggregate demand for the complex risky asset, I . Using the result in Proposition 4, these comparative statics also hold for α .

Proposition 5 *Using the market clearing condition, we have that the following comparative statics hold for aggregate investment in the complex risky asset in partial equilibrium:*

1. $\frac{\partial I}{\partial \sigma_v} < 0$
2. $\frac{\partial I}{\partial \gamma} < 0$
3. $\frac{\partial I}{\partial f_{xx}} < 0$.

Thus, in general equilibrium, we will have:

1. α is an increasing function of total risk
2. α is an increasing function of risk aversion
3. α is an increasing function of the maintenance cost.

In partial equilibrium, demand for the risky asset is decreasing in total volatility, risk aversion, and the maintenance cost. As a result, in general equilibrium α is increasing in total volatility, risk aversion, and the maintenance cost. An increase in these parameters proxies for greater asset complexity, and thus our model predicts that α will be higher in more complex asset markets in general equilibrium.

Investor-specific Sharpe ratios, Investor Participation, and Market-level Sharpe ratios With the analysis of equilibrium excess returns in hand, we now turn to the equilibrium risk-return tradeoff at the investor and market-level as described by the investor-specific, and market-level Sharpe ratios. We emphasize the variation across individual Sharpe ratios as a function of expertise; all investors face a common market clearing α , but their effective risk varies. For the market-level Sharpe ratio, two effects are present. First, there is the effect of any changes on parameters on the individual Sharpe ratios of participants. Second, there is a

selection effect, or the effect on participation. We provide an intuitively appealing condition, complementarity between expertise and complexity, under which participation declines as the asset becomes more complex. We focus on the equally weighted market-level equilibrium Sharpe ratio in our analysis. In addition to offering cleaner comparative statics because it does not depend on investor portfolio choices and market shares, the equally weighted Sharpe ratio represents the expected Sharpe ratio that an investor who could pay a cost to draw from the expertise distribution above the entry cutoff would earn. In that sense, it is the “expected Sharpe ratio”. Note that the Sharpe ratio for non-experts is not defined.

Investor-specific Sharpe ratios: We define the investor-specific Sharpe Ratio as:

$$SR(x) = \frac{\alpha}{\sigma(x)}.$$

We provide results for how investor-specific Sharpe ratios change as total volatility changes under the three possible cases for the elasticity of investor-specific risk with respect to total volatility in Proposition 6. The sign of this elasticity is a key determinant of our Sharpe ratio results.

Proposition 6 *The comparative statics for the investor-specific Sharpe ratios with respect to total volatility depend on whether expertise and complexity display complementarity. Specifically, the results depend on which of the following three possible cases for the elasticity of investor-specific risk with respect to total volatility, applies:*

1. **Case 1, Constant Elasticity:** If $\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}$ is a constant, that is

$$\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} = 0,$$

we must have that $SR(x)$ is either an increasing or a decreasing function of total risk for all expertise levels.

2. **Case 2, Increasing Elasticity:** If $\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}$ is an increasing function of expertise, that is

$$\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} > 0,$$

then there is a cutoff level x^ , such that for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} > 0$; and for all*

$x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} < 0$. Further, x^* exists if for any small $\varepsilon < 10^{-6}$

$$(0, \varepsilon) \subseteq \left\{ \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} \mid \text{for all } x \right\} \subseteq [0, \infty).$$

3. **Case 3, Decreasing Elasticity (complementarity):** If $\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}$ is a decreasing function of expertise, that is

$$\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0,$$

then there is a cutoff level x^* , such that for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} < 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} > 0$.

Since individual Sharpe ratios are ratios of excess returns relative to effective volatilities, variation in Sharpe ratios as total volatility changes depend on the elasticity of excess returns relative to the elasticities of effective volatilities as total volatility changes. This is intuitive, as these elasticities measure the equilibrium percentage change in excess returns vs. the percentage change in effective volatilities as underlying total risk varies. The change in α is aggregate, the same for all investors. So, the changes in individual Sharpe ratios with respect to changes in total volatility depend on the expertise-specific percentage changes in effective volatility relative to the percentage change in total volatility. Proposition 6 demonstrates that the effect of an increase in total volatility on individual Sharpe ratios varies in the cross section, except in the special case of Case 1. If the elasticity of effective volatilities as total volatilities changes is the same for all investors (Case 1), then the percentage change in α relative to the percentage change in effective volatility is the same for all investors. On the other hand, if the elasticity of effective volatilities with respect to total volatility is increasing in expertise (Case 2), then Sharpe ratios increase below a cutoff level of expertise and decrease above as total volatility increases. Finally, if this elasticity is declining in expertise, so that higher expertise investors face smaller increases in effective volatility as total volatility increases –expertise and complexity are complementary–, (Case 3), then Sharpe ratios increase above a cutoff level of expertise and decrease below. In this case, the economic rents from expertise are higher in more complex markets, characterized by higher total volatility.

We focus our analysis on Case 3, because it leads to the empirically plausible implication that more complex assets, with higher total volatilities, have *lower* participation despite having persistently elevated excess returns. Thus, we argue that the decreasing elasticity case is the most relevant for describing a long-run, stationary equilibrium in a complex asset market.

Moreover, it seems intuitive that the difference in effective volatilities between more and less complex assets would be smaller for higher expertise investors.

As a concrete example, consider the following two fixed income arbitrage strategies considered by Duarte, Longstaff, and Yu (2006). Mortgages are highly complex securities containing embedded prepayment options. MBS payoffs are affected by consumer behavior, house prices, and credit conditions, as well as interest rates. There is no agreed upon pricing model, and investors' strategy implementations vary widely as a result. By contrast, swap spread arbitrage follows a fairly straightforward long-short rule based on current LIBOR swap rates relative to Treasury yields and repo rates. The way this strategy is implemented is quite similar across investors. Accordingly, Duarte, Longstaff, and Yu (2006) show that mortgage related strategies (MBS) earn higher alphas, and Sharpe ratios, than simple swap spread arbitrage strategies. We argue, in agreement with their motivation and findings, that expertise is more valuable in MBS arbitrage. Put another way, the difference in the risk which investors face in MBS vs. Treasuries is decreasing in investor expertise. To be sure, highly sophisticated investors face more risk in MBS than in treasuries. However, the difference in effective risk across these two fixed income strategies is not as great for expert investors as it is for an inexperienced investor, consistent with Case 3 in Proposition 6.

Investor Participation Before analyzing market-level Sharpe ratios, we first describe investor participation. There are two key inputs into the market-level risk return tradeoff. First, incumbents' individual Sharpe ratios change. Second, as equilibrium α changes, participation also changes. This selection effect plays a key role in determining comparative static results in general equilibrium. We show in the Appendix that participation increases with total volatility in Cases 1 and 2 of Proposition 6. If all elasticities of $\sigma(x)$ with respect to σ_ν are the same, or if they are lower for lower expertise investors, then participation will increase with total volatility. This is intuitive because α must increase with total volatility σ_ν by enough for lower expertise investors to help to clear the market. Because it is intuitive that participation should decrease with asset complexity, we thus focus on results under Case 3 of Proposition 6. For this case, we provide a natural condition under which participation declines as the asset becomes more complex and total volatility increases in Proposition 7. The proof appears in the Online Appendix.

Proposition 7 *Define the entry cutoff \underline{x} as in Equation (18). Participation declines*

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} > 0$$

if the following conditions hold. Condition 1 is necessary. Condition 2 is sufficient.

1. Complementarity of Complexity and Expertise (Case 3 of Proposition 6):

$$\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0, \text{ and,}$$

2. Non-representativeness of Highest Elasticity Participants:

$$l_{\text{sup}}^{\sigma_\nu} > \left(1 + \frac{1}{1 + \mathcal{B}(\underline{x})}\right) E \left[\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} | x \geq \underline{x} \right],$$

where $l_{\text{sup}}^{\sigma_\nu}$ is defined to be the highest elasticity of all participating investors' effective volatility with respect to total volatility, and

$$\mathcal{B}(x) = \frac{2}{\beta(x)} \frac{\beta(x) + \gamma}{\beta(x) - 1}.$$

The first condition, namely that the elasticity of effective volatility with respect to total volatility is decreasing in expertise, is necessary for participation to decline as complexity, and total volatility, increase. To see this, consider the market clearing condition. For the market to clear without the participation by, and demand from, lower expertise investors as total volatility increases, it must be that higher expertise investors' demand is less adversely affected. The second condition gives a sufficient condition for participation to decrease if the first condition is satisfied. This condition states that the elasticity of the agent with the highest sensitivity of effective volatility to total volatility, which in Case 3 of Proposition 6 will be the lowest expertise agent who participates, must be sufficiently different from the average across participants, times a constant. Note that the constant will be near one if β is close to one, which it will be as it is the tail parameter from a Pareto distribution. The function $\mathcal{B}(x)$ gives the elasticity of the mean wealth at expertise level x with respect to total volatility. Intuitively, what is necessary for participation to decline as total volatility increases is that there is enough variation in the effect of the change in total volatility across agents with high and low expertise so that α does not need to increase enough to satisfy the marginal investor or to entice lower expertise investors to participate. Section 5.1 provides numerical examples, provides additional intuition, and illustrates that the parameter space for which the second condition is satisfied is large.

In sum of the results so far, under the conditions for complementarity between expertise and complexity in Proposition 7, our model generates higher persistent α 's and lower participation, despite free entry, as total volatility and asset complexity increase.

Equilibrium market-level Sharpe Ratio We define the equally weighted market equilibrium Sharpe ratio as:

$$SR^{ew} = E \left[\frac{\alpha}{\sigma(x)} \mid x \geq \underline{x} \right].$$

We focus on comparative statics for the equally weighted market equilibrium Sharpe ratio, however we note the following Corollary holds for the value weighted market Sharpe ratio:⁹

Corollary 2 *In Case 3 of Proposition 6, in which expertise and complexity are complementary, and in which participation declines with complexity, we have that an increase in the equally weighted Sharpe ratio as total volatility increases implies an increase in the value weighted Sharpe ratio.*

We omit the formal proof, which is straightforward. Intuitively, the value weighted Sharpe ratio weights higher expertise investors' Sharpe ratios more heavily, since they are wealthier and they allocate a greater fraction of wealth to the risky asset.

The following Proposition provides sufficient conditions under which SR^{ew} increases with total volatility σ_ν in Case 3 of Proposition 6. The other cases are presented in the Appendix, which also contains the proof. It is possible that the SR^{ew} increases with total volatility even if the increase impacts all investors equally, or more negatively impacts higher expertise investors. However, we emphasize that the joint observation of the SR^{ew} increasing *and* participation declining as total volatility changes is possible only in Case 3, in which expertise and complexity are complementary.

Proposition 8 *The equally weighted market Sharpe ratio is increasing with total risk in general equilibrium, i.e.,*

$$\frac{\partial SR^{ew}}{\partial \sigma_\nu} > 0,$$

If the following sufficient conditions are satisfied:

1. **Complementarity of Complexity and expertise (Case 3 of Proposition 6):**

$$\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0, \text{ and,}$$

2. **Non-representativeness of Marginal Participants:**

$$\underbrace{\frac{\partial [1 - \Lambda(\underline{x})] / [1 - \Lambda(\underline{x})]}{\partial \sigma_\nu/\sigma_\nu}}_{\text{elasticity of participation}} \overbrace{\left(1 - \frac{SR(\underline{x})}{SR^{ew}} \right)}^{\text{difference btw marginal SR and average}} > \underbrace{\frac{\partial SR(x) / SR(x)}{\partial \sigma_\nu/\sigma_\nu} \Big|_{x=\underline{x}}}_{\text{elasticity of marginal SR}} \quad (28)$$

⁹See the Appendix for the definition of the value-weighted market equilibrium Sharpe ratio.

The above conditions give us the sufficient conditions which yield an increasing Sharpe ratio when total volatility increases. We show an equivalent statement of the second condition, in which we substitute out the equilibrium Sharpe ratios with the underlying exogenous parameters and functions, in the Appendix. We use Equation (28) here to provide intuition. From Case 3 of Proposition 6, we have that individual Sharpe ratios increase above, and decrease below, a threshold level of participating expertise levels, $x^* > \underline{x}$, when total volatility increases. Thus, for the average Sharpe ratio to increase, the Sharpe ratios of participants with declining Sharpe ratios cannot be too important in determining the average. The greater holdings of higher expertise agents makes it more likely that the value weighted Sharpe ratio increases. For the equally weighted Sharpe ratio, participants above and below x^* are weighted according to the distribution of expertise. Thus, what is required is that there are not too many experts between \underline{x} and x^* , or that for these participants between \underline{x} and x^* , effective volatility does not increase by too much relative to the equilibrium average return α . If the Sharpe ratio does decline substantially for low expertise participants, the second condition ensures that the Sharpe ratio for these participants is not representative of the average over all participants. Equation (28) precisely formalizes this intuition: On the left hand side, the first term represents the elasticity of participation with respect to total volatility. In Case 3 of Proposition 6, participation is decreasing and so $\frac{\partial[1-\Lambda(\underline{x})]/[1-\Lambda(\underline{x})]}{\partial\sigma_\nu/\sigma_\nu}$ is negative, making this first term positive after applying the negative sign. The second term involves the relative value of the marginal Sharpe ratio to the population weighted average. It has a maximal value of one, and is closer to one when the Sharpe ratio of the lowest expertise agent is very small relative to the average Sharpe ratio. On the right hand side is the elasticity of the marginal Sharpe ratio with respect to total volatility. This term is always negative before applying the negative sign. Intuitively, on the right hand side is a force against the market Sharpe ratio increasing. If the marginal agent's Sharpe ratio is very adversely affected by the increase in total volatility, the effect on the market Sharpe ratio is strongly negative. On the left hand side, then, are two positive terms which offset this negative effect on the market Sharpe ratio by more the larger they are. The first term is large when a large mass of low expertise agents (with low Sharpe ratios) do not participate. The second term is large if the marginal agent is not representative of the market average; if for the marginal agent the decline in the Sharpe ratio is large but the effect is small for the average participant then the market Sharpe ratio will still increase. In short, this condition is easier to satisfy if: First, participation is very sensitive to total volatility, because then many low Sharpe ratio agents will drop out of the market. Second, if the difference between the marginal and average Sharpe ratios is large, because then the lower Sharpe ratios of the marginal agents are not representative, and finally if the marginal Sharpe ratio is not too sensitive to total volatility.

Section 5.1 provides numerical examples, provides additional intuition, and illustrates that the parameter space for which the second condition is satisfied is large.

In summary, we note that both for participation to decrease, and for the equilibrium Sharpe ratio to increase with asset complexity, similar conditions suffice. The first is that expertise and complexity are complementary. The second is that marginal participants are not representative of average market participants. This second condition ensures that markets can clear without low expertise agents, and that lower expertise participants with declining Sharpe ratios are not important for market equilibrium outcomes. Under these conditions, our model delivers a rational explanation for why more complex assets can generate a higher α , a higher equally-weighted equilibrium market Sharpe ratio, but have low participation, despite free entry. Intuitively, as in a standard industrial organization model, the superior volatility reduction technologies of more expert investors provide them with an excess of (risk-bearing) capacity, which serves to reduce the entry incentives of newcomers despite attractive conditions for incumbents.

5 Numerical Results: Cross Section and Time Series

This section presents some quantitative implications of our model.

5.1 Numerical Comparative Statics: Cross Section Results

We provide additional intuition for our analytical results, and develop new results for wealth concentration across assets with different complexity. We focus on a functional form for effective volatility which satisfies the condition in Case 3 from Proposition 6, in which the elasticity of effective volatility with respect to total volatility declines with expertise.¹⁰ Specifically, and notationally reintroducing the dependence of $\sigma(x)$ on total volatility σ_ν , we specify that:

$$\sigma^2(\sigma_\nu, x) = a + x^{-b}\sigma_\nu^2. \quad (29)$$

This function is increasing in total volatility at a rate that decreases with expertise, i.e. it satisfies:

$$\frac{\partial \sigma(\sigma_\nu, x)}{\partial \sigma_\nu} > 0,$$

and,

$$\frac{\partial \frac{\partial \sigma(\sigma_\nu, x) / \sigma(\sigma_\nu, x)}{\partial \sigma_\nu / \sigma_\nu}}{\partial x} < 0.$$

¹⁰Results for the other cases are available upon request.

Numerical Comparative Statics: Asset Pricing and Participation We begin with simple, illustrative numerical comparative statics for our main asset pricing and participation results. The model generates closed form policy functions and wealth distributions conditional on expertise levels. To provide intuition for the effects of equilibrium pricing, we provide the comparative statics in both partial equilibrium and general equilibrium. In partial equilibrium, the excess return is given exogenously, and held fixed, while aggregate demand (and hence implicitly supply) varies. In general equilibrium, the excess return is computed endogenously given a fixed (normalized) supply of the risky asset. Because α and I form a bijection (Proposition 4 provides conditions for which they are one to one and onto), for any given supply of the complex risky asset, we can find the unique solution to the fixed point problem determining α through market clearing. The Appendix describes the solution algorithm.

Our baseline parameters are summarized in Table 3. The time interval is one quarter. The risk-free rate, subjective discount factor, and the maintenance cost are all set to 1%. The coefficient of relative risk-aversion is 5. The total standard deviation of the risky asset return is 40%. We set $a = 0.16\%$ and $b = 1.5$. This implies that the highest expertise investors can eliminate 83% of total risk, and face an effective standard deviation of 6.8%. The marginal expertise investors at the entry cutoff can eliminate 45% of total risk, and face an effective standard deviation of 22.1%, which is about half that of total volatility. The minimum normalized wealth is set to 1%. The log-normal distribution of expertise has a mean of 0 and volatility of 3 before truncation. We choose 500 grid points for expertise levels between the lowest and highest levels of expertise. The lowest expertise level is pinned down by Equation (18) for participation. We choose a value of 10,000 for the highest expertise level. The value ensures that the policy functions for the most expert investors very closely approximate the limiting case of infinite expertise, and that there is no mass point at the upper bound. The density of expertise on the grid points is then chosen to approximate a truncated log-normal distribution.

First, we provide intuition for the sufficient Condition 2 in Proposition 7, which requires non-representativeness of the highest elasticity participants, and the sufficient Condition 2 in Proposition 8, which requires non-representativeness of marginal participants. When these two sufficient conditions are satisfied, along with the necessary condition for participation to decline (Condition 1 in Proposition 7 and Proposition 8, requiring complementarity between complexity and expertise), the model generates lower participation, higher excess returns, and higher Sharpe ratios for more complex assets. It is intuitive that high expertise investors face smaller differences in Sharpe ratios between simple and complex strategies and assets than do low expertise investors, i.e. that complexity and expertise are complementary. Figure 2 shows

Table 3: Numerical Example: Baseline Parameter Values

Parameter	Symbol	Value	Target
Discount factor	ρ	0.01	Annual interest rate
Risk-free rate	r_f	0.01	Annual interest rate
Risk aversion	γ	5	
Entry cost	fnx	0.03	
Maintenance cost	fix	0.01	
Supply of risky asset	S	1.20	Average annual excess return $\alpha = 6.98\%$
Risky assets growth rate	g_S	3%	
Total volatility of risky asset return	σ_ν	40%	
Mean of expertise distribution	μ_x	0	
Volatility of expertise distribution	σ_x	3	
Constant in $\sigma^2(x)$	a	0.16%	Average SR
Slope of $\sigma^2(x)$	b	1.5	Dispersion in effective volatilities
Minimum normalized wealth	z_{\min}	0.01	

parameters for which the sufficient conditions in Proposition 7 and Proposition 8 are satisfied. This figure plots model comparative statics showing when these conditions are satisfied given variation in the dispersion in expertise (y-axis) and the curvature of effective volatility in expertise (x-axis). The parameter b in $\sigma^2(\sigma_\nu, x) = a + x^{-b}\sigma_\nu^2$ describes the curvature of effective volatility. The dispersion in the (truncated) log-normal distribution of expertise, x , is given on the y-axis. These two parameters are the key inputs to determining the representativeness of high elasticity, marginal, participants, since the curvature parameter determines how fast effective volatility declines with expertise, and the dispersion in the expertise distribution determines the relative mass of investors with different levels of expertise. All other parameters are held at the baseline values in Table 3. The top left panel displays blue for parameters for which the sufficient Condition 2 in Proposition 7, which requires non-representativeness of the highest elasticity participants, is satisfied, and red for parameters for which it is violated. The top right panel uses the same color scheme to show that participation is decreasing at whenever this sufficient condition is satisfied. The bottom left panel displays blue for parameters for which the sufficient Condition 2 in Proposition 7, which requires non-representativeness of the highest elasticity participants, is satisfied, and red for parameters for which it is violated. The bottom left panel uses the same color scheme to show that the equilibrium equally-weighted Sharpe ratio is increasing whenever this sufficient condition is satisfied.

The figure shows that these two conditions are satisfied for most reasonable parameter combinations. As dispersion in expertise decreases, and effective volatility becomes closer to linear (b decreases), excess returns become implausibly high. At the boundary for participation to be increasing, the smallest α is 38%. This is because when dispersion in expertise is small, or curvature of effective volatility is low, the total demand from high expertise investors is smaller, implying that α must be large enough to compensate the low expertise agents to participate. It is intuitive that Condition 2 of Propositions 7 and Proposition 8 are less likely to be satisfied when b or σ_x is small. Both lower curvature in effective volatility, and lower dispersion in expertise, make it more likely that the marginal participant is more similar to the average participant. Moreover, when b is small, effective volatility is closer to displaying a constant elasticity of effective volatility with respect to total volatility. In other words, as b gets small, the elasticities of effective volatility with respect to total volatility are closer to satisfying the constant elasticity Case 1 of Proposition 7, for which participation is increasing. The parameter combinations for which the two conditions are satisfied are very similar. The top right and bottom left panels of Figure 2 together show that Condition 2 of Proposition 8 is satisfied whenever participation is decreasing. Finally, note that the Sharpe ratio appears to be always increasing in these example economies which satisfy Case 3 of Proposition 7.

With the intuition for the parameters which yield large enough differences between marginal agents and other participants in hand, we turn to a numerical example to illustrate our main results for the cross section. Figure 3 displays the effects of changes in total volatility, with more complex assets characterized by higher total volatility. Starting in the top row, as total volatility increases, demand for the risky asset in partial equilibrium decreases, implying a higher α in general equilibrium. The left hand side of the second row displays the entry cut-off, which is increasing in fundamental volatility, consistent with our result in Proposition 7. Accordingly, participation, graphed on the right hand side of the second row, declines. We note that participation declines by less in general equilibrium, due to the positive effect of total volatility on α , but still the decline is nearly as large as in partial equilibrium given our parametric assumptions. Finally, the third row plots the equally weighted standard deviation of the risky asset returns, which are increasing in both partial and general equilibrium. In partial equilibrium, α is held constant, so participation declines substantially as total volatility increases. The selection effect then significantly attenuates the increase in the equally weighted effective volatility. By contrast, in general equilibrium the increase in α compensates for the increase in total risk somewhat, participation declines by less, and so the equally weighted effective volatility increases by more than in partial equilibrium. Finally, the bottom right panel of Figure 3 shows that despite the fact that the equally weighted standard deviation is

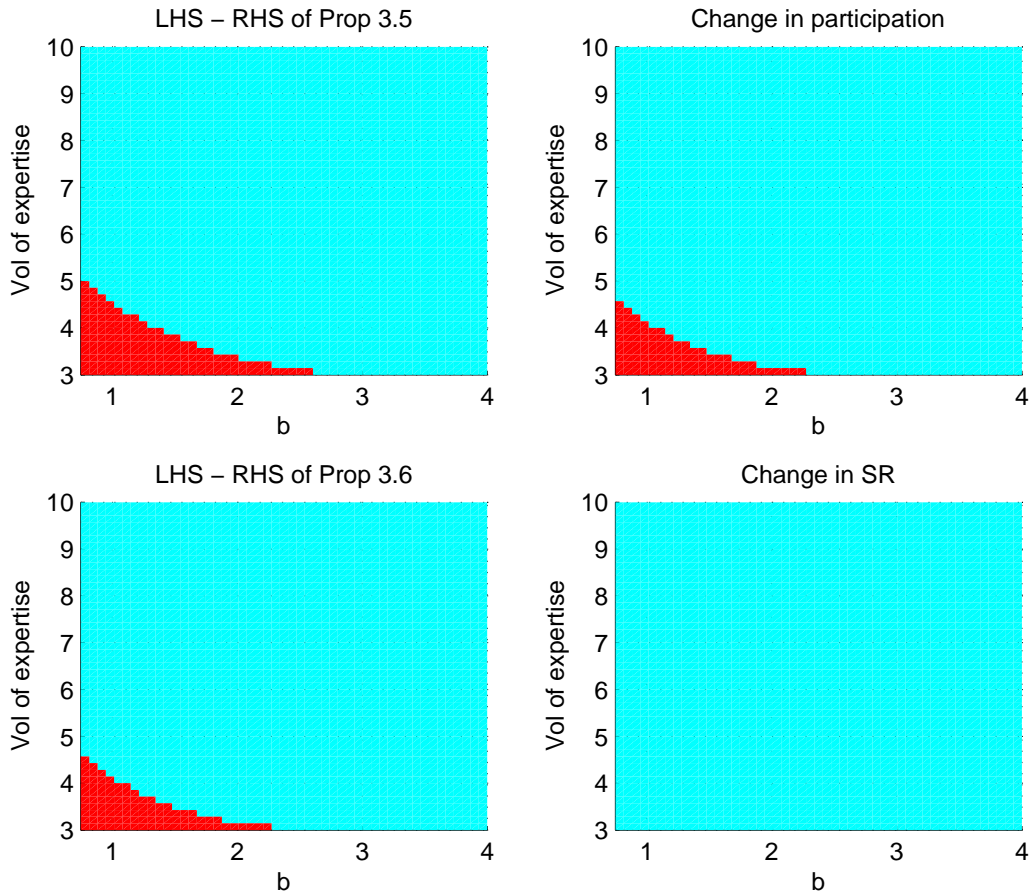


Figure 2: Model comparative statics over variation in the dispersion in expertise (y-axis) and the curvature of effective volatility in expertise (x-axis). Economies satisfy Case 3 of Proposition 6, i.e. expertise and complexity are complementary, using the functional form $\sigma^2(\sigma_\nu, x) = a + x^{-b}\sigma_\nu^2$ for effective volatility, and a (truncated) log-normal distribution for x with a mean of zero and dispersion given on the y-axis. The top left panel displays blue for parameters for which the sufficient Condition 2 in Proposition 7, which requires non-representativeness of the highest elasticity participants, is satisfied, and red for parameters for which it is violated. The top right panel uses the same color scheme to show that participation is decreasing whenever this condition is satisfied. The bottom left panel displays blue for parameters for which the sufficient Condition 2 in Proposition 8, which requires non-representativeness of marginal participants, is satisfied, and red for parameters for which it is violated. The bottom right panel uses the same color scheme to show that the equilibrium equally-weighted Sharpe ratio is increasing whenever this condition is satisfied.

increasing, the larger, positive effect of the increase in α in general equilibrium implies that the equally weighted Sharpe ratio increases, consistent with Proposition 8. Thus, the numerical example confirms the model’s ability to generate persistently higher α ’s and larger Sharpe ratios, but lower participation despite free entry, for more complex assets characterized by higher total volatility, consistent with the empirical findings in Pontiff (1996) and Duarte, Longstaff, and Yu (2006).

5.2 Numerical Results: Transition Dynamics

Finally, we explore the dynamic impact of a large increase in idiosyncratic risk on markets for asset-backed loans. This shock will induce significant exit on the part of experts and reduce the risk-bearing capacity in this market, leading to large and persistent dislocations that contribute to large increase in spreads.

Return Dynamics in MBS During Financial Crisis To set the stage, we take a close look at return dynamics in MBS markets during the crisis. Figure 4 plots a 12-month moving average of the cross-sectional and time-series volatility measure, realized returns, mean AUM (in logs), the standard deviation of AUM (in logs) and finally, the Gini coefficient in the MBS segment of the hedge fund industry. The run-up in idiosyncratic volatility was accompanied by a large drop in realized returns (-2.5% per month at the nadir), and a subsequent recovery. At the same time, the mean AUM declined substantially at the onset of the crisis.

Table 4 reports the moments of the monthly *realized alpha*, the implied Sharpe ratio based on the cross-sectional volatility measure, and the implied Sharpe ratio based on the time-series vol measure. For each fund, the alphas are computed by running regressions of the fund’s excess returns on excess returns on the Bloomberg Barclays Treasury index and the Bloomberg Barclays US MBS Index, and 3 lags of the regressors, to allow for illiquidity and slow marking-to-market in hedge fund returns (Assness, Krail, and Liew, 2001). We then compute the monthly cross-sectional averages of fund-level forward-looking alpha; the value α_t is the cross-sectional average of the fund-level intercepts computed for the $[t + 23]$ window.

Panel A reports the results for the universe of funds investing in MBS excluding multi-asset funds. The monthly alpha increases from 0.36% before the crisis to an average 1.02% during the crisis (2007.6-2009.6). During the crisis, there was a steep increase in realized monthly alpha from -1.72% at the start to 1.14% at the end of the crisis. After the crisis, the alpha reverts back to its pre-crisis value.

In spite of the the large increases in volatility, the compensation per unit of risk increases during the crisis. After the initial decline in realized alpha at the start of the crisis, we see

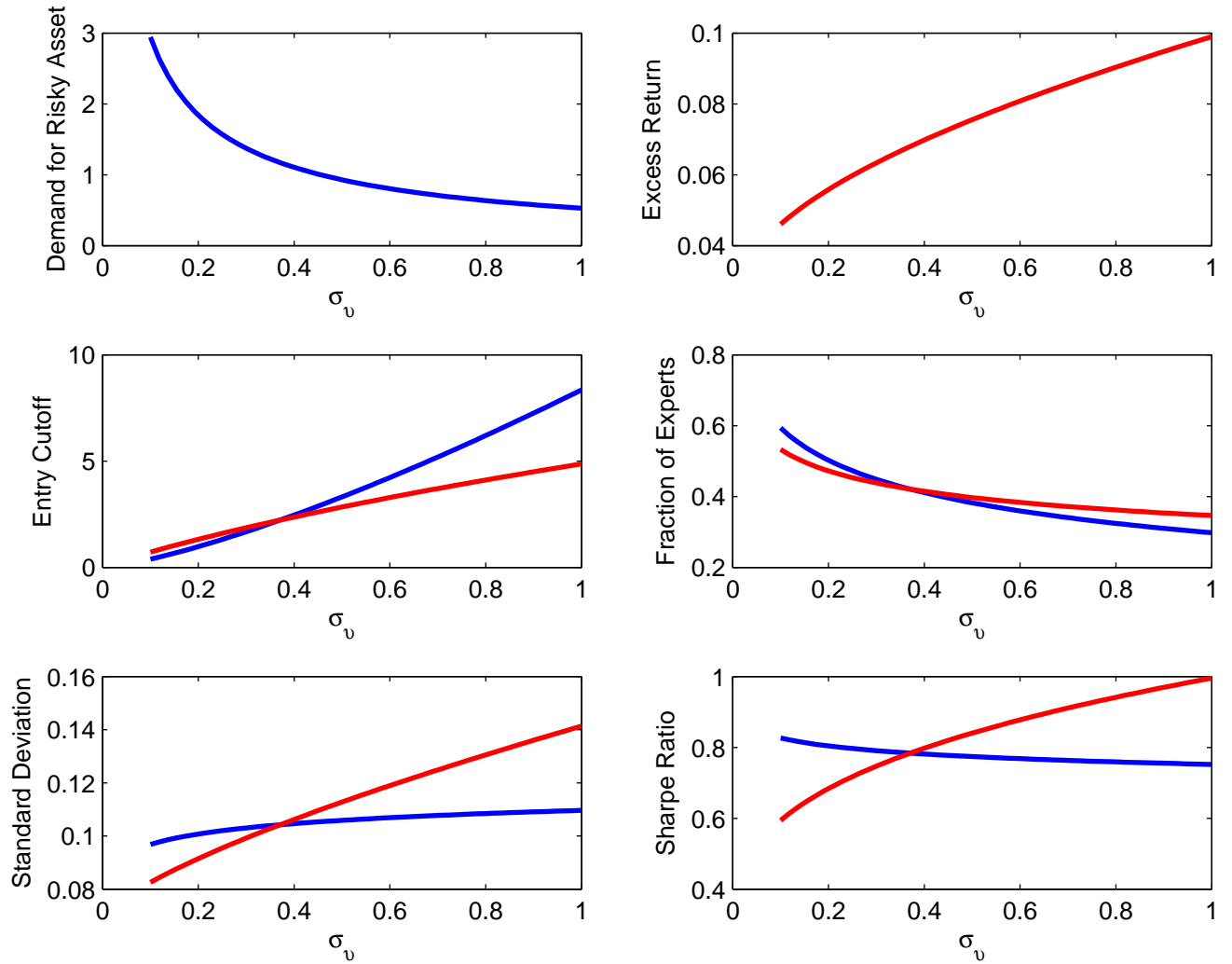


Figure 3: Model comparative statics over variation in total risk. Blue lines plot partial equilibrium comparative statics, red lines plot general equilibrium comparative statics. Economies satisfy Case 3 of Proposition 6, i.e. expertise and complexity are complementary.

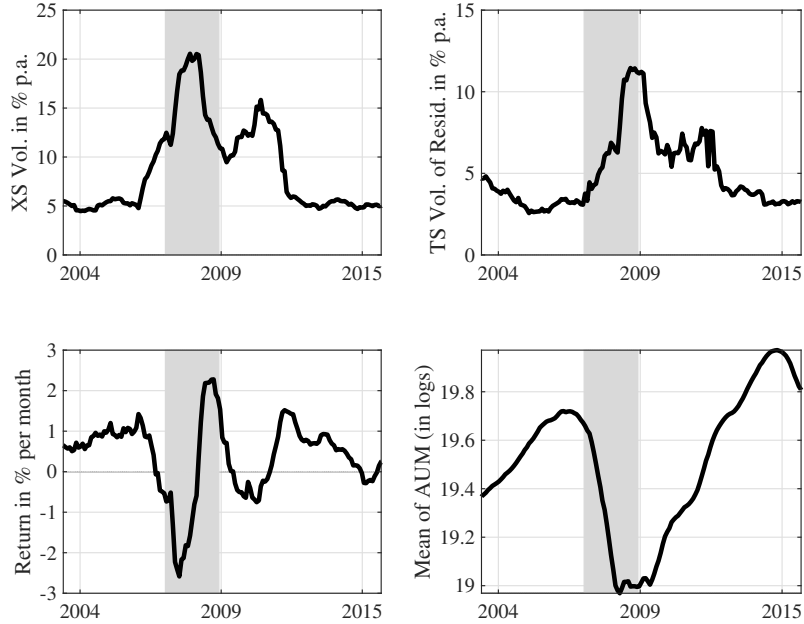


Figure 4: Vol and Return Dynamics of MBS Funds. The shaded areas represent the U.S. financial crisis. Sample excludes multi-asset funds. 12-month moving average of XS Vol, TS Vol, Value-weighted returns (in % per month), mean AUM (in logs), Std. of AUM (in logs) and Gini of AUM.

immediate and large increases in the Sharpe ratio during the crisis. We use two measures of the Sharpe ratio. The XS and TS Sharpe ratios are computed as $\alpha/XS\ Vol$ and $\alpha/TS\ Vol$ respectively. The annualized XS SR increases from an average pre-crisis value of 0.51 to an average of 0.76 during the financial crisis. Towards the end of the crisis, the SR peaks at 0.89. The TS SR measure displays similar dynamics.¹¹ Finally, Panel B and C report the results for the universe excluding multi-asset funds and funds not based in the U.S., and for all funds investing in MBS respectively. The stylized facts are quite similar for all three samples.

Algorithm for Computing Transition Dynamics To gauge the effects of such an unanticipated increase in total fund-specific volatility, we increase the σ_v parameter in the effective volatility equation (29), and then trace out the dynamics of all endogenous variables to this unanticipated shock. Specifically, we assume that at $t = 0$, investors wake up to realize that volatility will be higher for 10 quarters. Starting at $t = 9$, σ_v reverts to its calibrated values in Table 3. We solve the transition dynamics using the following reverse-shooting algorithm. Guess an initial path of $\alpha_0(t)$, $t = 0, 1, 2, 3, 4, \dots$ and then for $l = 0, 1, 2, \dots$,

¹¹The TS SR is higher overall, because the TS Vol is lower than XS Vol.

Table 4: Realized MBS Hedge Fund Alpha. Returns in Asset-Backed Fixed Income. Sample: Monthly data from 1991.12-2018.08. The crisis is defined as the period from 2007.6-2009.6. We report the averages for the pre-crisis, crisis, and post-crisis periods, as well as the 12-month average prior to the start and the end of the crisis. Cross-sectional average of realized fund-level α in % per month, and the annualized (realized) Sharpe ratio computed as $\alpha/XS Vol$ and $\alpha/TS Vol$. Alpha computed for each fund by regressing excess fund return on excess returns on the Bloomberg Barclays Treasury index and the Bloomberg Barclays US MBS Index in 24-month rolling windows. We include 3 lags of the regressors. Value at t α_t is cross-sectional average of intercepts for $[t + 23]$ window. Source: Thomson-Reuters Lipper Hedge Fund Database.

	Pre-crisis		Crisis		Post-Crisis
	<i>Average</i>	<i>Start</i>	<i>Average</i>	<i>End</i>	<i>Average</i>
Panel A: Excluding Multi-Asset Funds					
Alpha (% per month)	0.36	-1.72	1.02	1.14	0.34
XS SR (p.a.)	0.51	-1.27	0.76	0.89	0.56
TS SR (p.a.)	0.78	-1.84	1.25	2.04	1.29
Panel B: Excluding Multi-Asset and Funds without U.S. Focus					
Alpha (% per month)	0.43	-0.80	1.36	1.58	0.39
XS SR (p.a.)	0.66	-1.87	1.24	1.66	0.73
TS SR (p.a.)	0.92	-0.89	1.82	3.54	1.39
Panel C: Including All MBS Funds					
Alpha (% per month)	0.31	-1.77	0.77	1.09	0.22
XS SR (p.a.)	0.47	-1.76	0.67	0.95	0.38
TS SR (p.a.)	0.72	-2.15	1.08	2.43	0.87

1. Given $\alpha_l(t)$, compute the endogenous wealth loss at 0 implied by the path of α :

$$\Delta w_0 = \theta(x, 0) \int_0^\infty \exp(-\rho t) (\alpha_l(t) - \alpha_{ss}) dt.$$

2. Given $\alpha_l(t)$, solve HJB equation with terminal condition $V_{new}^x(w, x, t)$ backward in time to compute time path of value functions $V_l^x(w, x, t)$ and policy functions $\theta_l(x, t)$.
3. Given the policy function, solve the Kolmogorov Forward equation with initial distribution of wealth and expertise forward to calculate the time path of wealth distributions $\phi_l^x(w, x, t)$.
4. Given $\theta_l(x, t)$ and $\phi_l^x(w, x, t)$ calculate

$$S_l(t) = \int \int \lambda(x) \phi_l^x(w, x, t) \theta_l(x, t) w(x, t) dw dx.$$

5. Update $\alpha_{l+1}(t) = \alpha_l(t) + \kappa(S - S^l(t))$, where $\kappa > 0$. That is, if demand is less than total supply, we increase excess returns $\alpha_l(t)$ to clear the market.
6. Stop when $\alpha_{l+1}(t)$ is sufficiently close to $\alpha_l(t)$.

Impulse Response to Background Risk Shock We use the calibrated version of the model from the previous section. Table 3 reports the calibrated parameters. Figure 5 plots the transition path for an unanticipated shock that roughly doubles the effective volatility for 10 quarters. In this experiment, total volatility before expertise is applied, σ_ν , is raised by 200% for 10 quarters. The average effective vol is 5.8% before the crisis. In response to the shock, the average effective vol (panel 1) increases to 8% during the crisis. Although the increase in total volatility is large, the increase in risk is mainly absorbed by the better experts; the worst experts exit the market (extensive margin), or reduce their exposure to the risky asset (intensive margin) in response to the shock. As a result, the increase in effective volatility is much smaller. Although the measured increase in volatility in the MBS hedge fund industry was larger, our model is still able to generate substantial changes in alpha, average Sharpe ratios, and participation. We conjecture that the average level of expertise may have also fallen during the crisis, since prepayment behavior changed considerably. Reducing the mean of expertise, or (equivalently) the average effect of expertise on effective volatility, is an additional lever with which to match the larger observed effects of the crisis, however we focus on changing only one parameter to provide cleaner intuition for our main mechanism.

The alpha (shown in panel 2) increases from 3% to more than 4.5% per annum to induce the experts to absorb all this additional risk, while the equally-weighted Sharpe ratio increases from 0.58 to 0.67. The total wealth of participants (panel 4) initially declines by 20%, partly because the price of the asset declines, causing (equally weighted) average participant wealth to decline (panel 5) and partly because participation declines (panel 6). Experts close to the margin decide to leave after the increase in volatility (panel 7). The cumulative exit rate for experts is 18% during those 10 quarters. However, after that, total expert wealth quickly recovers because the incumbents rebuild wealth, taking advantage of high alpha. The fraction of investors active as experts declines by more than 10 percentage points initially, and only starts to recover after 10 quarters. The response along the participation margin is critical to the model’s ability to deliver response like the one observed during the crisis. Leverage (equally-weighted average, shown in panel 8) initially increases, because the higher SR induces the remaining experts to lever up and clear the market for the risky asset, but after that, the lower alpha induces a persistent decrease in leverage.

As experts quickly rebuild wealth, by taking advantage of the elevated alpha, the risk-bearing capacity of the experts recovers; the alphas and Sharpe ratios start to decrease immediately after the shock. When volatility declines to its original level after 10 quarters, alpha actually decreases slightly below its steady-state level, because expert wealth is higher than the steady-state level. As a result, it takes much longer for leverage (panel 8) to recover to its steady-state value, even after volatility reverts back. While our model produces dynamics that are qualitatively similar to the data, the variation in MBS alpha, SRs and volatility are somewhat smaller than what was observed during the crisis. This is partly because our model cannot fully replicate the increase in volatility in MBS returns in the data without changing additional parameters.

Endogenous entry is key to replicating the increase in the Sharpe ratio. To gauge the importance of endogenizing entry/exit, we set the fixed cost of participation to zero: $f_{xx} = 0$, and plot the response to the same shock in Figure 6 . There is no entry or exit in response to the shock. The effective increase in vol is much larger now, because we have eliminated exit. All investors are now participants. In this case, the equal-weighted Sharpe ratio declines in response to the vol shock, which is clearly counterfactual. It is critical to model entry/exit in order to match the risk price dynamics in the data.

Interestingly, an exogenous shock to expert wealth (a common negative return shock) produces counterfactual effects, because its main effect is to induce entry of more experts. Figure 7 plots the impulse responses to a 20% destruction of expert wealth. Without a shock to volatility, wealth shocks cannot replicate the dynamics in the data. The wealth shock also produces

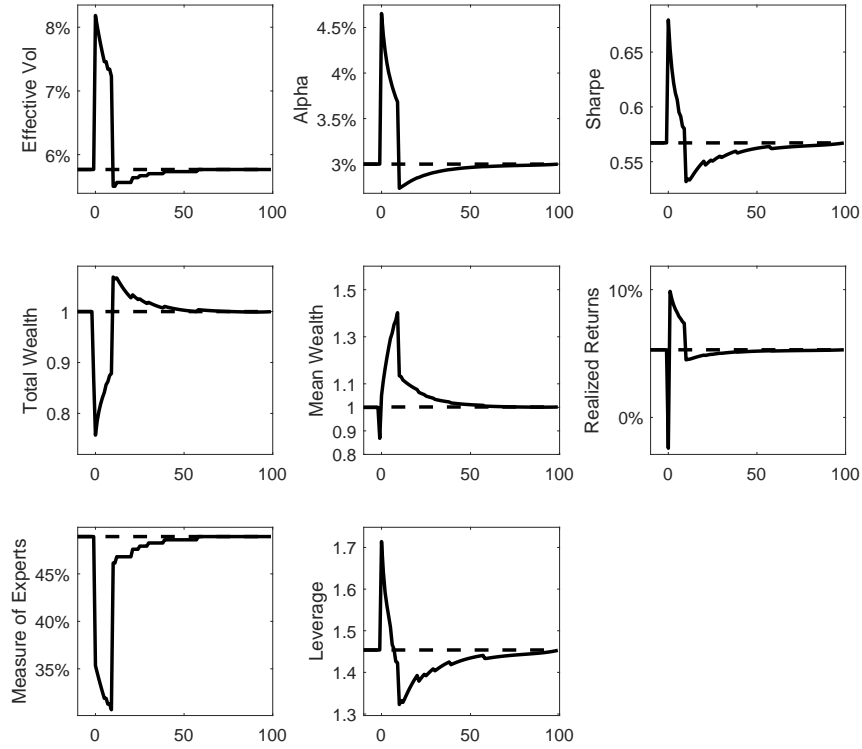


Figure 5: Impulse response to volatility shock. Experiment: σ_v increases by 200% for 10 quarters. Effective vol (panel 1): equally weighted average effective volatility among all participants; Alpha (panel 2): excess returns of risky assets; Sharpe ratio (panel 3): equally weighted Sharpe ratio among all participants; Total wealth (panel 4): aggregate wealth held by participants (normalized to 1 as compared to SS); mean wealth (panel 5): average wealth of all participants (normalized to 1 as compared to SS); measure of experts (panel 6): ratio of market participants over total population; leverage (panel 7): equally weighted average leverage of investment on risky asset (panel 8).

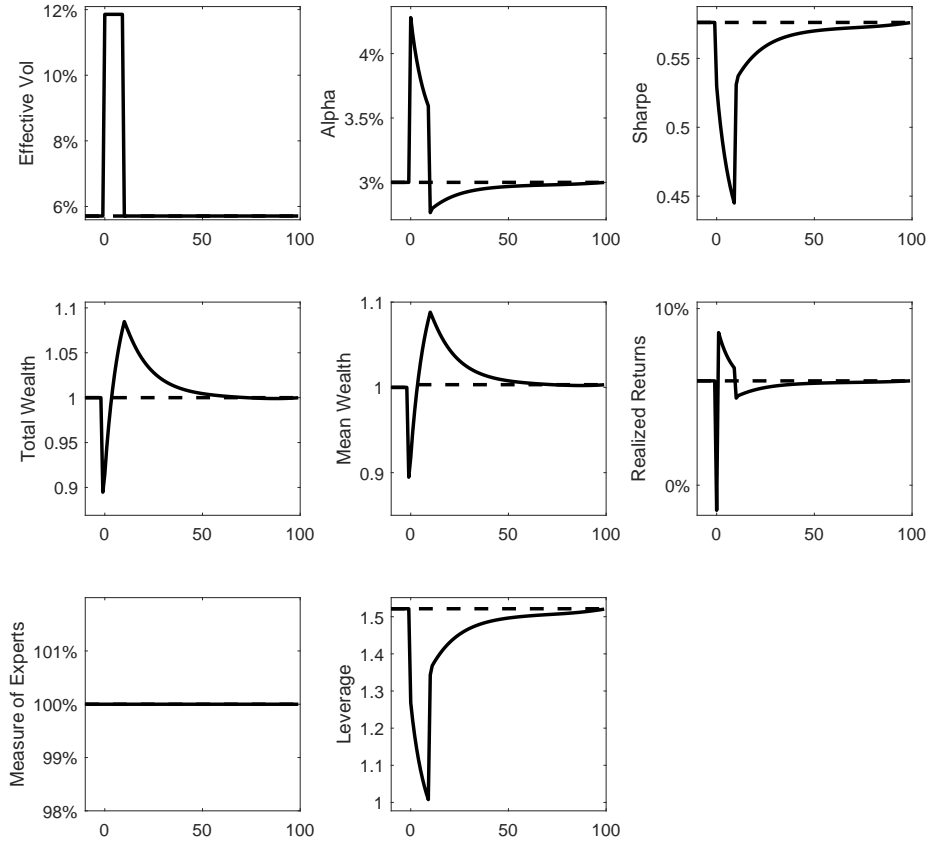


Figure 6: Impulse response to volatility shock. No Entry/Exit ($f_{xx} = 0$). Experiment: σ_ν increases by 200% for 10 quarters. Effective vol (panel 1): equally weighted average effective volatility among all participants; Alpha (panel 2): excess returns of risky assets; Sharpe ratio (panel 3): equally weighted Sharpe ratio among all participants; Total wealth (panel 4): aggregate wealth held by participants (normalized to 1 as compared to SS); mean wealth (panel 5): average wealth of all participants (normalized to 1 as compared to SS); measure of experts (panel 6): ratio of market participants over total population; leverage (panel 7): equally weighted average leverage of investment on risky asset (panel 8).

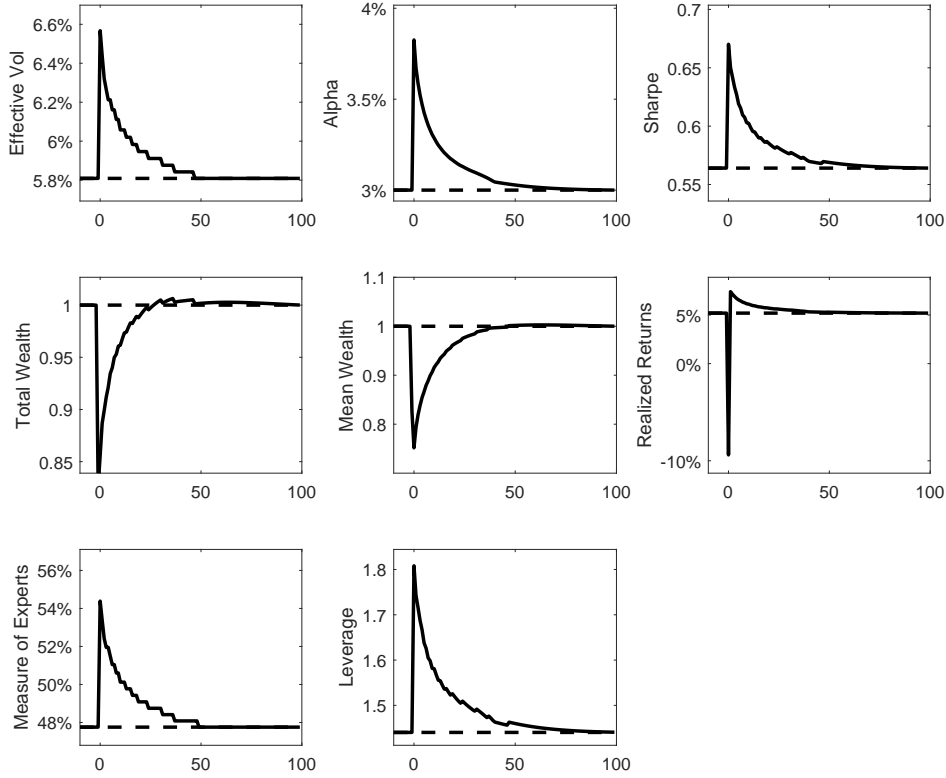


Figure 7: Impulse response to wealth shock. Experiment: expert wealth declines by 20%. Effective vol (panel 1): equally weighted average effective volatility among all participants; Alpha (panel 2): excess returns of risky assets; Sharpe ratio (panel 3): equally weighted Sharpe ratio among all participants; Total wealth (panel 4): aggregate wealth held by participants (normalized to 1 as compared to SS); mean wealth (panel 5): average wealth of all participants (normalized to 1 as compared to SS); measure of experts (panel 6): ratio of market participants over total population; leverage (panel 7): equally weighted average leverage of investment on risky asset (panel 8).

a counter-factual increase in leverage in the aftermath of the crisis (panel 8).

6 Conclusion

We study the equilibrium properties of complex asset markets. A complex asset is defined as an investment which requires a model and implementation strategy, thereby exposing investors to idiosyncratic risk. We provide a specific derivation of how complex assets impose idiosyncratic risk on investors. Investors attempt to harvest alpha by going long a fundamental asset with some alpha, and short an imperfect tracking portfolio in their long-short strategies designed to maximize returns while minimizing risk. We provide evidence that idiosyncratic risk is an

important component of risk in complex assets, and one that appears to increase substantially in times of market turmoil.

In our equilibrium model, required returns increase with asset complexity, as proxied for by higher total investor-specific volatility. We emphasize heterogeneity in the risk-return tradeoff faced by investors with different levels of expertise. Accordingly, we show that in our model, if expertise and complexity are complementary, improvements in market-level Sharpe ratios can be accompanied by lower market participation, consistent with empirical observations. Finally, we describe the transition dynamics after a shock to volatility in our model. Following an increase in total volatility, the economy displays an elevated alpha, an increase in the average Sharpe ratio, and a decline in participation. We show that these patterns match those seen in the MBS hedge fund industry, a prototypical example of a complex asset market.

References

- ACHARYA, V. V., H. S. SHIN, AND T. YORULMAZER (2009): “A Theory of Slow-Moving Capital and Contagion,” (7147).
- ACHDOU, Y., J. HAN, J.-M. LASRY, P.-L. LIONS, AND B. MOLL (2014): “Heterogeneous Agent Models in Continuous Time,” *Working Paper*.
- ADRIAN, T., AND N. BOYARCHENKO (2013): “Intermediary Leverage Cycles and Financial Stability,” *Working Paper FRB NY*.
- ADRIAN, T., E. ETULA, AND T. MUIR (2014): “Financial Intermediaries and the Cross-Section of Asset Returns,” *The Journal of Finance*, 69(6), 2557–2596.
- AIYAGARI, S., AND M. GERTLER (1999): ““Overreaction” of Asset Prices in General Equilibrium,” *Review of Economic Dynamics*, 2, 3–35.
- ALLEN, F., AND D. GALE (2005): “From Cash-in-the-Market Pricing to Financial Fragility,” *Journal of the European Economic Association*, 3, 535–546.
- ASSNESS, C., R. KRAIL, AND J. LIEW (2001): “Do hedge funds Hedge?,” *Journal of Portfolio Management*, 28(1), 6–19.
- BASAK, S., AND B. CROITORU (2000): “Equilibrium Mispricing in a Capital Market with Portfolio Constraints,” *Review of Financial Studies*, 13, 715–748.
- BASAK, S., AND D. CUOCO (1998): “An Equilibrium Model with Restricted Stock Market Participation,” *Review of Financial Studies*, 11, 309–341.
- BENHABIB, J., A. BISIN, AND S. ZHU (2011): “The Distribution of Wealth and Fiscal Policy in Economies With Finitely Lived Agents,” *Econometrica*, 79(1), 123–157.
- (2015): “The Wealth Distribution in Bewley Economies with Capital Income Risk,” *Journal of Economic Theory*, 159, Part A, 489 – 515.
- (2016): “The Distribution of Wealth in the Blanchard-Yaari Model,” *Macroeconomic Dynamics*, pp. 1–16.
- BEWLEY, T. (1986): “Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers,” in *Contributions to Mathematical Economics in Honor of Gerard Debreu*, ed. by W. Hildenbrand, and A. Mas-Collel. North-Holland, Amsterdam.
- BOROVICKA, J. (2019): “Survival and Long-run Dynamics with Heterogeneous Beliefs Under Recursive Preferences,” *J. Polit. Econ.*
- BRENNAN, M., AND E. SCHWARTZ (1990): “Arbitrage in Stock Index Futures,” *Journal of Business*, 63, s7–s31.

- CHIEN, Y., H. COLE, AND H. LUSTIG (2011): “A Multiplier Approach to Understanding the Macro Implications of Household Finance,” *Review of Economic Studies*, 78, 199–234.
- CHIEN, Y., H. COLE, AND H. LUSTIG (2012): “Is the Volatility of the Market Price of Risk Due to Intermittent Portfolio Rebalancing?,” *American Economic Review*, 102(6), 2859–96.
- CLEMENTI, G. L., AND B. PALAZZO (2016): “Entry, Exit, Firm Dynamics, and Aggregate Fluctuations,” *American Economic Journal: Macroeconomics*, 8(3), 1–41.
- DERMAN, E. (2016): “Bloomberg Masters in Business podcast,” .
- DIEP, P., A. L. EISFELDT, AND S. A. RICHARDSON (2017): “Prepayment risk and expected MBS returns,” .
- DITELLA, S. (2016): “Uncertainty Shocks and Balance Sheet Recessions,” *Journal of Political Economy*.
- DIXIT, A., AND R. PINDYCK (1994): *Investment Under Uncertainty*. Princeton University Press.
- DRECHSLER, I. (2014): “Risk Choice under High-Water Marks,” *Review of Financial Studies*, 27, 2052–2096.
- DUARTE, J., F. A. LONGSTAFF, AND F. YU (2006): “Risk and Return in Fixed-Income Arbitrage: Nickels in Front of a Steamroller?,” *Review of Financial Studies*, 20(3), 769–811.
- DUFFIE, D. (2010): “Presidential Address: Asset Price Dynamics with Slow-Moving Capital,” *The Journal of Finance*, 65(4), 1237–1267.
- DUFFIE, D., AND B. STRULOVICI (2012): “Capital Mobility and Asset Pricing,” *Econometrica*, 80, 2469–2509.
- DUMAS, B. (1989): “Two-person Dynamic Equilibrium in the Capital Market,” *Review of Financial Studies*, 2, 157–188.
- EDMOND, C., AND P.-O. WEILL (2012): “Aggregate Implications of Micro Asset Market Segmentation,” *Journal of Monetary Economics*, pp. 319–335.
- FROOT, K. A., AND P. G. J. O’CONNELL (1999): *The Financing of Catastrophe Risk* chap. The Pricing of U.S. Catastrophe Reinsurance, pp. 195–232. National Bureau of Economic Research.
- FUNG, W., AND D. A. HSIEH (1997): “Empirical Characteristics of Dynamic Trading Strategies: The Case of Hedge Funds,” *Review of Financial Studies*, 10, 275–302.
- FUNG, W., D. A. HSIEH, N. Y. NAIK, AND T. RAMADORAI (2008): “Hedge Funds: Performance, Risk, and Capital Formation,” *The Journal of Finance*, 63(4), 1777–1803.
- GABAIX, X. (2009): “Power Laws in Economics and Finance,” *Annual Review of Economics*.

- GABAIX, X., A. KRISHNAMURTHY, AND O. VIGNERON (2007): “Limits of Arbitrage: Theory and Evidence from the Mortgage-Backed Securities Market,” *Journal of Finance*, 62, 557–595.
- GABAIX, X., J.-M. LASRY, P.-L. LIONS, AND B. MOLL (2016): “The Dynamics of Inequality,” *Econometrica*, 84(6), 2071–2111.
- GABAIX, Z. (1999): “Zipf’s Law for Cities: An Explanation,” *Quarterly Journal of Economics*, August, 739–767.
- GĂRLEANU, N., S. PANAGEAS, AND J. YU (2015): “Financial Entanglement: A Theory of Incomplete Integration, Leverage, Crashes, and Contagion,” *American Economic Review*, 105(7), 1979–2010.
- GLODE, V., R. C. GREEN, AND R. LOWERY (2012): “Financial Expertise as an Arms Race,” *The Journal of Finance*, 67(5), 1723–1759.
- GOETZMANN, W. N., J. E. INGERSOLL, AND S. A. ROSS (2003): “High-Water Marks and Hedge Fund Management Contracts,” *Journal of Finance*, 58, 1685–1717.
- GOURIO, F., AND N. ROYS (2014): “Size-Dependent Regulations, Firm Size Distribution, and Reallocation,” *Quantitative Economics*, 5, 1759–7331.
- GREENWOOD, R. (2011): “Investor Demand: Module Note for Instructors,” *Harvard Business School Publishing*, 5-211-101.
- GROMB, D., AND D. VAYANOS (2002): “Equilibrium and Welfare in Markets with Financially Constrained Arbitrageurs,” *Journal of Financial Economics*, 66, 361–407.
- (2010a): “The Limits of Arbitrage: the State of the Theory,” *Annual Review of Financial Economics*, 2, 251–275.
- (2010b): “A Model of Financial Market Liquidity Based on Arbitrageur Capital,” *Journal of the European Economic Association Papers and Proceedings*, pp. 456–466.
- HADDAD, V. (2014): “Concentrated Ownership and Equilibrium Asset Prices,” *Working Paper*.
- HARRISON, J. M. (2013): *Brownian Models of Performance and Control*. Cambridge University Press.
- HE, Z., B. KELLY, AND A. MANELA (2017): “Intermediary asset pricing: New evidence from many asset classes,” *Journal of Financial Economics*, 126(1), 1–35.
- HE, Z., AND A. KRISHNAMURTHY (2012): “A Model of Capital and Crises,” *Review of Economic Studies*, 79, 735–777.
- (2013): “Intermediary Asset Pricing,” *American Economic Review*, 103(2), 732–70.

- HOMBERT, J., AND D. THESMAR (2014): “Overcoming Limits of Arbitrage: Theory and Evidence,” *Journal of Financial Economics*, 111(1), 26 – 44.
- HOPENHAYN, H. A. (1992a): “Entry, Exit, and firm Dynamics in Long Run Equilibrium,” *Econometrica*, 60(5), 1127–1150.
- (1992b): “Exit, Selection, and the Value of Firms,” *Journal of Economic Dynamics and Control*, 16, 621–653.
- KACPERCZYK, M., J. B. NOSAL, AND L. STEVENS (2014): “Investor Sophistication and Capital Income Inequality,” *Working Paper*.
- KAPLAN, G., B. MOLL, AND G. L. VIOLANTE (2016): “Monetary Policy According to HANK,” *NBER Working Paper*, (21897).
- KARLIN, S., AND H. M. TAYLOR (1981): *A Second Course in Stochastic Processes*. Academic Press.
- KOGAN, L., S. A. ROSS, J. WANG, AND M. M. WESTERFIELD (2006): “The Price Impact and Survival of Irrational Traders,” *J. Finance*, 61(1), 195–229.
- KOGAN, L., AND R. UPPAL (2001): “Risk Aversion and Optimal Portfolio Policies in Partial and General Equilibrium Economies,” *Working Paper*.
- KONDOR, P. (2009): “Risk in Dynamic Arbitrage: The Price Effects of Convergence Trading,” *Journal of Finance*, 64, 631–655.
- KONDOR, P., AND D. VAYANOS (2014): “Liquidity Risk and the Dynamics of Arbitrage Capital,” *Working Paper, LSE*.
- KRUEGER, D., AND H. LUSTIG (2009): “When is Market Incompleteness Irrelevant for the Price of Aggregate Risk (and when is it not)?,” *Journal of Economic Theory*.
- KURLAT, P. (2016): “Asset Markets With Heterogeneous Information,” *Econometrica*, 84(1), 33–85.
- LEE, D. W., AND T. S. KIM (2014): “Idiosyncratic Risk and Cross Section of Hedge Fund Returns,” *Working Paper*.
- LUTTMER, E. G. (2007): “Selection, Growth, and the Size Distribution of Firms,” *Quarterly Journal of Economics*, 122, 1103–1144.
- MERTON, R. C. (1987): “A Simple Model of Capital Market Equilibrium with Incomplete Information,” *The Journal of Finance*, 42(3), 483–510.
- MIAO, J. (2005): “Optimal Capital Structure and Industry Dynamics,” *Journal of Finance*, 60, 2621–2659.
- MITCHELL, M., L. H. PEDERSEN, AND T. PULVINO (2007): “Slow Moving Capital,” *American Economic Review*, 97, 215–220.

- MITCHELL, M., AND T. PULVINO (2012): “Arbitrage Crashes and the Speed of Capital,” *Journal of Financial Economics*, 104, 469–490.
- MOLL, B. (2014): “Productivity Losses from Financial Frictions: Can Self-Financing Undo Capital Misallocation?,” *American Economic Review*, 104(10), 3186–3221.
- MUIR, T. (2014): “Financial Crises and Risk Premia,” *Working Paper, Yale University*.
- PANAGEAS, S., AND M. WESTERFIELD (2009): “High-Water Marks: High Risk Appetites? Convex Compensation, Long Horizons, and Portfolio Choice,” *Journal of Finance*, 64, 1–36.
- PASQUARIELLO, P. (2014): “Financial Market Dislocations,” *Review of Financial Studies*.
- PLANTIN, G. (2009): “Learning by Holding and Liquidity,” *Review of Economic Studies*, 76, 395–412.
- PONTIFF, J. (1996): “Costly Arbitrage: Evidence from Closed-End Funds,” *Quarterly Journal of Economics*.
- PONTIFF, J. (2006): “Costly Arbitrage and the Myth of Idiosyncratic Risk,” *Journal of Accounting and Economics*, 42, 35–52.
- SANDRONI, A. (2000): “Do Markets Favor Agents able to Make Accurate Predictions?,” *Econometrica*, 68(6), 1303–1341.
- SHARPE, W. F. (1966): “Mutual Fund Performance,” *Journal of Business*, 39, 119–138.
- SHLEIFER, A., AND R. W. VISHNY (1997): “The Limits of Arbitrage,” *The Journal of Finance*, 52(1), 35–55.
- TITMAN, S., AND C. TIU (2011): “Do the Best Hedge Funds Hedge?,” *Review of Financial Studies*, 24, 123–168.
- TODA, A. A. (2014): “Incomplete Market Dynamics and Cross-Sectional Distributions,” *Journal of Economic Theory*.
- TODA, A. A., AND K. WALSH (2015): “The Double Power Law in Consumption and Implications for Testing Euler Equations,” *Journal of Political Economy*.
- XIONG, W. (2001): “Convergence Trading with Wealth Effects: An Amplification Mechanism in Financial Markets,” *Journal of Financial Economics*, 62, 247–292.
- YUAN, K. (2005): “Asymmetric Price Movements and Borrowing Constraints: A Rational Expectations Equilibrium Model of Crises, Contagion, and Confusion,” *Journal of Finance*, 60, 379–411.

All Appendix Material to be Posted Online Only This Online Appendix contains two sections. The first section contains the proofs to the formal statements in the main text. The second section contains additional results on a economy with fund flows related to expertise, the efficiency of the joint wealth expertise distribution, an economy with two complex assets, and alternative models for investor heterogeneity (participation costs, risk aversion).

A Appendix: Proofs

A.1 Proofs for Section 3

Lemma A.1 Policy Functions and Prices Independent of Aggregate Shocks. *The derivation of the return process in Equation (1) provided in Derivation 1, along with Assumption 2, are sufficient to ensure that neither policy functions nor equilibrium prices are functions of the realization of the fundamental shock $dB^F(t)$.*

Proof. Lemma A.1 First, since the innovations in $dB^F(t)$ are independent over time, the realizations do not affect portfolio choices or consumption decisions. Moreover, Lemma 1 describes how fundamental shocks are absorbed into the investor-specific risk $\sigma(x)dB_i(t)$ in Equation 1, which implies that the proof for our Proposition 1 applies as stated. Then, for asset prices to be independent of the realization of the fundamental shock $dB^F(t)$, all that remains is to show that the expected growth rate of wealth, conditional on each expertise level, is the same under the return process stated in (1) and the return process constructed in Lemma 1.

We can rewrite the expected growth rate of wealth as

$$\begin{aligned} \frac{dw(t)}{w(t)} &= \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1)\alpha^2(t)}{2\gamma^2\sigma^2(x)} \right) dt + \frac{\alpha(t)}{\gamma\sigma(x)} dB(t) \\ &= \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1)\alpha^2(t)}{2\gamma^2\sigma^2(x)} \right) dt + \frac{\alpha(t)}{\gamma\sigma(x)} [(1 - \rho_i(x))\sigma^F dB^F(t) + \sigma^T(x)dB^T(t)]. \end{aligned}$$

Second, because $\frac{\rho_o(x) + \rho_u(x)}{2} = 1$, we have that

$$\sigma_o(x) = \sigma_u(x) = \sqrt{[(1 - \rho_o(x))\sigma^F]^2 + [\sigma^T(x)]^2} = \sqrt{[(1 - \rho_u(x))\sigma^F]^2 + [\sigma^T(x)]^2}.$$

Thus, each expert's expected growth rate of wealth, conditional on the aggregate shock is given by:

$$E \left[\frac{dw(t)}{w(t)} | x, dB^F(t) \right] = \frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1)\alpha^2(t)}{2\gamma^2\sigma^2(x)} + \frac{\alpha(t)}{\gamma\sigma(x)} (1 - \rho_i(x))\sigma^F dB^F(t).$$

If there is a positive aggregate shock, the growth rate of the wealth of under-hedging investors, with lower ρ , is higher than the growth rate of wealth for over-hedging investors, with higher ρ , and vice versa. The difference is exactly canceled out at the expertise-level aggregate. That is:

$$E \left[\frac{dw(t)}{w(t)} | x \right] = \frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1)\alpha^2(t)}{2\gamma^2\sigma^2(x)}.$$

Aggregating across expertise levels, then, the equilibrium excess return α does not depend on the realization of aggregate shock. ■

Proof. Proposition 1. Value Functions Homogenous in Wealth. We prove this Proposition by guess and verify. We assume that the following transversality condition holds, which can be satisfied

by assuming an upper bound on the short position in the risk free asset that is non-binding for the agents with the highest level of expertise.

$$\lim_{T \rightarrow \infty} E[e^{-\rho T} V^x(W_T)] = 0.$$

First, we write the HJB equations of our model

$$\begin{aligned} \max_{c^x(x,t), \theta(x,t)} 0 &= u(c^x(x,t)) + V_w^x [w(t)(r_f + \theta(x,t)\alpha(t)) - c^x(x,t) - f_{xx}w(t)] \\ &\quad + \frac{\theta^2(x)\sigma^2(x)w(t)^2}{2} V_{ww}^x - \rho V^x \\ \max_{c^n(t)} 0 &= u(c^n(t)) + V_w^n (r_f w(t) - c^n(t)) - \rho V^n \end{aligned}$$

The first order conditions for optimality are given by:

$$\begin{aligned} u'(c^x(x,t)) &= V_w^x, \\ u'(c^n(t)) &= V_w^n, \\ V_w^x \alpha(t) + \theta(x,t)\sigma^2(x)w(t)V_{ww}^x &= 0. \end{aligned}$$

Next, we guess that the value functions have the following form:

$$\begin{aligned} V^x(w(t), x) &= y^x(x,t) \frac{w(t)^{1-\gamma}}{1-\gamma}, \\ V^n(w(t), x) &= y^n(t) \frac{w(t)^{1-\gamma}}{1-\gamma}. \end{aligned}$$

Given these conjectures, it follows from the Benveniste-Scheinkman condition that the optimal consumption choices are given by:

$$\begin{aligned} c^x(x,t) &= [y^x(x,t)]^{-\frac{1}{\gamma}} w(t), \\ c^n(t) &= [y^n(t)]^{-\frac{1}{\gamma}} w(t), \end{aligned}$$

and that the optimal portfolio choice is given by

$$\theta(x,t) = \frac{\alpha(t)}{\gamma\sigma^2(x)}.$$

Plugging these choices into the HJB equations, we get

$$\begin{aligned} 0 &= [y^x(x,t)]^{-\frac{1-\gamma}{\gamma}} + y^x(x,t) \left(r_f + \frac{\alpha^2(t)}{\gamma\sigma^2(x)} - [y^x(x,t)]^{-\frac{1}{\gamma}} - f_{xx} \right) (1-\gamma) \\ &\quad - \frac{\alpha^2(t)}{2\gamma\sigma^2(x)} y^x(x,t) (1-\gamma) - \rho y^x(x,t) \\ &= \gamma [y^x(x,t)]^{-\frac{1-\gamma}{\gamma}} + y^x(x,t) \left(r_f + \frac{\alpha^2(t)}{2\gamma\sigma^2(x)} - f_{xx} \right) (1-\gamma) - \rho y^x(x,t), \\ 0 &= \gamma [y^n(t)]^{-\frac{1-\gamma}{\gamma}} + y^n(t) (1-\gamma) r_f - \rho y^n(t). \end{aligned}$$

Rearranging the equations, we solve for $y^x(x,t)$ and $y^n(x,t)$,

$$\begin{aligned} y^x(x,t) &= \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha^2(t)}{2\gamma^2\sigma^2(x)} \right]^{-\gamma}, \\ y^n(t) &= \left[\frac{(\gamma-1)r_f + \rho}{\gamma} \right]^{-\gamma}. \end{aligned}$$

Given all policy functions, we get the experts' wealth growth rates:

$$\frac{dw(t)}{w(t)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1) \alpha^2(t)}{2\gamma^2 \sigma^2(x)} \right) dt + \frac{\alpha(t)}{\gamma \sigma(x)} dB(t)$$

Finally, given homogeneity of the value functions in wealth, the participation cutoff is constructed by direct comparison between $y^x(x, t)$ and $y^n(t)$. ■

Proof. Proposition 2. Equivalence of value and policy functions under the reflecting barrier z_{\min} .

Interpretation of z_{\min} : We assume that one of two things can happen to an investor at the minimum fund size, z_{\min} . With probability q , the investor's fund is liquidated and the investor is replaced with a new agent with normalized wealth z_{\min} and the same expertise as the exiting agent. Note that liquidation would ceterus paribus cause the incumbent agent to be conservative, to avoid z_{\min} . However, with probability $1 - q$, the agent receives a cash infusion which allows them to remain in the market with normalized wealth equal to z_{\min} . We implement the cash infusion using a reflected Brownian motion. Note that this cash infusion in isolation would cause the agent to risk shift, to take advantage of limited liability at z_{\min} . Intuitively, we require that $E[V^x(z, x)] = qE[V^{\text{liquidation}}] + (1 - q)E[V^{\text{rescue}}]$, conditional on the optimal policies under the true normalized wealth dynamics. Since the value under the true, non-reflecting, dynamics lies between the value of liquidation and the value of rescue, we conjecture (and verify below) that there exists some probability, conditional on parameters, that this is the case. It seems realistic that investors face uncertainty about what will happen to them if their assets reach a lower threshold. Will they be liquidated, or be able to raise new funds? Note that our proof offers a technical contribution for models with endogenous state variables following a reflecting geometric Brownian motion.

We show that the optimal policies in the model with reflecting barrier z_{\min} are equivalent to those in the original model under our assumptions for probabilities and values of liquidation vs. rescue. Our proof assumes an optimal voluntary exit date. This is without loss of generality in a stationary equilibrium with no entry or exit.

The model with geometric Brownian motion normalized wealth dynamics is given by the true Model 1 as follows:

$$\begin{aligned} V^x(z(t), x) &= \max_{c^x(x,t), T, \theta^x(x,t)} \mathbb{E} \left[\int_t^T e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(T-t)} V^n(z(t)) \right] \\ \text{s.t. } dz(t) &= [z(t)(r_f + \theta^x(x, t) \alpha(t)) - c^x(x, t) - F_{xx}] dt \\ &\quad + z(t) \theta^x(x, t) \sigma(x) dB(t), \end{aligned}$$

The model with reflected Brownian motion normalized wealth dynamics is given by the alternative Model 2 as follows:

$$\begin{aligned} V^r(z(t), x) &= \max_{c^r(x,t), T, \theta^r(x,t)} \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x, s)) ds + e^{-\rho(s'-t)} \left[(1 - q) V^r(z_{\min}, x) \right. \right. \\ &\quad \left. \left. + q V^n(z(t), x) \right] \right] \\ \text{s.t. } dz(t) &= [z(t)(r_f + \theta^r(x, t) \alpha(t)) - c^r(x, t) - F_{xx}] dt \\ &\quad + z(t) \theta^r(x, t) \sigma(x) dB(t), \end{aligned}$$

where $s' < T$ is the first time that the agent's wealth falls below z_{\min} , the superscript n denotes the value of a non-expert which cannot re-enter, and the superscript r denotes the value of an expert

under reflected Brownian dynamics, given by $dz_t/z_t = \mu_z dt + \sigma_z dB_t$ for $z_t > z_{\min}$ and $dz_t/z_t = \max(\mu_z dt + \sigma_z dB_t, 0)$ for $z_t \leq z_{\min}$.¹²

For parsimony, we only present the case in which the expert normalized wealth drops below z_{\min} before the voluntary exit stopping time at which an expert would voluntarily choose to exit, which applies in the stationary equilibrium with exit time $T = \infty$ since there is no voluntary entry or exit in equilibrium.

To equate value and policy functions under the true and reflected normalized wealth dynamics then requires the appropriate specification of q , the probability of liquidation vs. receiving new funds which allow the investor to remain in the complex asset market with a normalized wealth of z_{\min} . Two alternative specifications for q both lead to equal values and policies. The first specifies that the value upon liquidation is the value of a non-expert which cannot re-enter, which is the same for all levels of expertise. In this case, the probability of liquidation must decline with expertise. The second alternative specifies that the value of liquidation is increasing in expertise. In this case, the probability of liquidation is the same for all agents. In what follows, we use the case in which the value upon exit is the value of a non-expert.¹³ Define

$$q(z(t), x) = \frac{y^x(x) - \left[\frac{z_{\min}}{z(t)}\right]^{1-\gamma}}{y^n(x) - y^x(x) \left[\frac{z_{\min}}{z(t)}\right]^{1-\gamma}}, \text{ for } z(t) \leq z_{\min}. \quad (\text{A.1})$$

Using this definition and equation (10) from Proposition 1, it is straightforward to show that

$$V^x(z(t), x) = (1 - q)V^x(z_{\min}, x) + qV^n(z(t)), \text{ for } z(t) \leq z_{\min}. \quad (\text{A.2})$$

Then, it is sufficient to show that

$$V^r(z(t), x) = V^x(z(t), x), \text{ for all } x \text{ and } z(t) \geq z_{\min}.$$

Our proof strategy is to first show that the value function and optimal policy functions are identical when expert wealth equals z_{\min} . Next, we show that the two models are identical for $z > z_{\min}$.

First, in Model 2, $c^r(x, t)$ is the optimal consumption, therefore,

$$\begin{aligned} V^r(z_{\min}, x) &= \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x, s)) ds + e^{-\rho(s'-t)} [(1 - q)V^r(z_{\min}, x) + qV^n(z(t'))] \right] \\ &\geq \mathbb{E} \left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(s''-t)} [(1 - q)V^r(z_{\min}, x) + qV^n(z(t''))] \right], \end{aligned}$$

where on the right hand side of the inequality we replace c^r with c^x and allow the time s' at which the normalized wealth declines to z_{\min} to update to s'' accordingly. Rearranging terms, we have:

$$\begin{aligned} V^r(z_{\min}, x) &= \frac{1}{1 - E[(1 - q)e^{-\rho(s'-t)}]} \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x, s)) ds + e^{-\rho(s'-t)} qV^n(z(t')) \right] \\ &\geq \frac{1}{1 - E[(1 - q)e^{-\rho(s''-t)}]} \mathbb{E} \left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(s''-t)} qV^n(z(t'')) \right]. \end{aligned}$$

Second, in Model 1, we can rewrite the value of being an expert as the value of the stream of con-

¹²See Harrison (2013), who also notes that the “reflected geometric Brownian motion process” might more precisely be called a regulated Brownian motion.

¹³The alternative is to specify q to be independent of the level of expertise, and given by $q(z(t), x) = \frac{1 - \left[\frac{z_{\min}}{z(t)}\right]^{1-\gamma}}{y^r - \left[\frac{z_{\min}}{z(t)}\right]^{1-\gamma}}$. In this case, the value of exit must be specified to be proportional to the expertise-specific value, i.e. $V_{exit}^r(z(t), x) = y^r V^x(z(t), x)$, where $y^r > 1$.

sumption before s'' plus the continuation value if wealth falls below z_{\min} . We have:

$$\begin{aligned}
V^x(z_{\min}, x) &= \mathbb{E} \left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(s''-t)} V^x(z(t''), x) \right] \\
&= \mathbb{E} \left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(s''-t)} [(1-q)V^x(z_{\min}, x) + qV^n(z(t''))] \right] \\
&\geq \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x, s)) ds + e^{-\rho(s'-t)} [(1-q)V^x(z_{\min}, x) + qV^n(z(t'))] \right],
\end{aligned}$$

where the second equality uses the result in Equation (A.2), and the inequality results from replacing c^x with c^r and allowing the expected time s'' at which the normalized wealth declines to z_{\min} under the policy c^r to update to s' accordingly. Rearranging terms, we have

$$\begin{aligned}
V^x(z_{\min}, x) &= \frac{1}{1 - E[(1-q)e^{-\rho(s''-t)}]} \mathbb{E} \left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(s''-t)} qV^n(z(t)) \right] \\
&\geq \frac{1}{1 - E[(1-q)e^{-\rho(s'-t)}]} \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x, s)) ds + e^{-\rho(s'-t)} qV^n(z(t)) \right].
\end{aligned}$$

We have established two inequalities which hold in opposite directions. Thus, we must have equality, namely:

$$\begin{aligned}
&\frac{1}{1 - E[(1-q)e^{-\rho(s''-t)}]} \mathbb{E} \left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(s''-t)} qV^n(z(t)) \right] \\
&= \frac{1}{1 - E[(1-q)e^{-\rho(s'-t)}]} \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x, s)) ds + e^{-\rho(s'-t)} qV^n(z(t)) \right],
\end{aligned}$$

and

$$V^r(z_{\min}, x) = V^x(z_{\min}, x).$$

Next, we show that the value functions for Model 1 and Model 2 are identical when $z > z_{\min}$. Using analogous logic, we have:

$$\begin{aligned}
V^r(z(t), x) &= \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x, s)) ds + e^{-\rho(s'-t)} [(1-q)V^r(z_{\min}, x) + qV^n(z(t'))] \right] \\
&\geq \mathbb{E} \left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(s''-t)} [(1-q)V^x(z_{\min}, x) + qV^n(z(t''))] \right] \\
&= \mathbb{E} \left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(s''-t)} V^x(z(t''), x) \right] \\
&= V^x(z(t), x), \text{ for all } z(t)
\end{aligned}$$

with equality iff $c^x(x, t) = c^r(x, t)$ and $\theta^x(x, t) = \theta^r(x, t)$. Also, we have:

$$\begin{aligned}
V^x(z(t), x) &= \left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(s''-t)} [(1-q)V^x(z_{\min}, x) + qV^n(z(t''))] \right] \\
&\geq \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x, s)) ds + e^{-\rho(s'-t)} [(1-q)V^r(z_{\min}, x) + qV^n(z(t'))] \right] \\
&= V^r(z(t), x), \text{ for all } z(t)
\end{aligned}$$

with equality iff $c^x(x, t) = c^r(x, t)$ and $\theta^x(x, t) = \theta^r(x, t)$. Therefore, our definition of the probabilities for liquidation vs. rescue in Equation (A.1) yields equivalence for all value and policy functions under the true and reflected dynamics models:

$$\begin{aligned} V^x(z(t), x) &= V^r(z(t), x), \text{ for all } x \text{ and } z(t) \\ c^x(x, t) &= c^r(x, t), \text{ for all } x \text{ and } z(t) \\ \theta^x(x, t) &= \theta^r(x, t), \text{ for all } x \text{ and } z(t). \end{aligned}$$

Our proof should enable more researchers to use reflecting dynamics to ensure stationarity in models with endogenous policies, especially if analytically tractable Pareto distributions are desired. In the model of Gabaix (1999), which introduces the method of using reflecting dynamics to generate a stationary Pareto distribution, cities do not choose size. This is in contrast to models such as ours with endogenous state variables.

Proof. Proposition 3 Tail Parameter Guess and Verify. We prove this Proposition by guess-and-verify. We guess that the stationary distribution takes the following form:

$$\phi(z, x) = C(x)z^{-\beta(x)-1},$$

Then, by plugging this guess into the Kolmogorov forward equation, we obtain the following condition:

$$\begin{aligned} 0 &= -\partial_z \left(z^{-\beta(x)} \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1)\alpha^2}{2\gamma^2\sigma^2(x)} - g_S \right) \right) \\ &\quad + \frac{1}{2}\partial_{zz} \left(z^{1-\beta(x)} \frac{\alpha^2}{\gamma^2\sigma^2(x)} \right) \\ &= \beta(x) \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1)\alpha^2}{2\gamma^2\sigma^2(x)} - g_S \right) \\ &\quad - \frac{1}{2}\beta(x)(1 - \beta(x)) \left[\frac{\alpha}{\gamma\sigma(x)} \right]^2 \\ &= \beta(x) \left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2(\gamma + \beta(x))}{2\gamma^2\sigma^2(x)} - g_S \right] \end{aligned}$$

Thus, by collecting terms, we obtain:

$$\begin{aligned} \beta(x) &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\ C_1 &= 2\gamma(f_{xx} + \rho - r_f + \gamma g_S), \\ C(x) &= \frac{1}{\int z^{-\beta-1} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma} \Big|_{z_{\min}}. \end{aligned}$$

Note there are two roots of equation

$$0 = \beta(x) \left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2(\gamma + \beta(x))}{2\gamma^2\sigma^2(x)} - g_S \right].$$

The negative drift of normalized wealth ensures that there will be one root of this equation which is larger than one. We then take this root in order to ensure that the mean wealth has a finite mean. ■

Proof. Corollary 1. Tail Parameter Highest Expertise. For the highest expertise agents, we

have

$$\begin{aligned}\bar{z} &= \int_{z_{\min}}^{\infty} z \phi(z, \bar{x}) dz = \int_{z_{\min}}^{\infty} C(\bar{x}) z^{-\beta(\bar{x})} dz \\ &= \frac{1}{-\beta(\bar{x}) - 1} C(\bar{x}) z^{-\beta(\bar{x}) - 1} \Big|_{z_{\min}}^{\infty} = z_{\min} \left[1 + \frac{1}{\beta(\bar{x}) - 1} \right].\end{aligned}$$

This gives us another expression for $\beta(\bar{x})$,

$$\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}}.$$

Also, we know that the decay coefficient is given by:

$$\beta(\bar{x}) = 2\gamma(f_{xx} + \rho - r_f + \gamma g_S) \frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma$$

Therefore, by combining these expressions, we have

$$2\gamma(f_{xx} + \rho - r_f + \gamma g_S) \frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma = \frac{1}{1 - z_{\min}/\bar{z}},$$

By rearranging the above equation, we get the following expression:

$$g_S = \frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2}{2\gamma\sigma^2(\bar{x})} + \frac{\alpha^2}{2\gamma^2\sigma^2(\bar{x})} \frac{1}{1 - z_{\min}/\bar{z}}.$$

We plug g_S into $\beta(x)$, to obtain:

$$\beta(x) = \left(\gamma + \frac{z_{\min}/\bar{z}}{1 - z_{\min}/\bar{z}} \right) \frac{\sigma^2(x)}{\sigma^2(\bar{x})} - \gamma.$$

■

A.2 Proofs for Section 4

Proof. Lemma 2 Comparative Statics: Expertise-Level Investment Direct calculation. We use 1 to denote a positive sign. For ease of exposition, we study the partial derivative of the log of expertise level investment in the risky asset. The sign of the partial derivative of the log and the level will be the same because log is a positive transformation. We have:

$$\begin{aligned}\log I(x) &= \log \frac{\alpha}{\gamma\sigma^2(x)} + \log Z(x) \\ &= \log \alpha - \log \gamma - \log \sigma^2(x) + \log Z(x),\end{aligned}$$

where $Z(x)$ is the total expertise level normalized wealth,

$$\begin{aligned}Z(x) &= \int_{z_{\min}}^{\infty} z \phi(z, x) dz = \int_{z_{\min}}^{\infty} C(x) z^{-\beta(x)} dz \\ &= \frac{1}{-\beta(x) - 1} C(x) z^{-\beta(x) - 1} \Big|_{z_{\min}}^{\infty} = z_{\min} \left[1 + \frac{1}{\beta(x) - 1} \right].\end{aligned}$$

Then:

1.

$$\begin{aligned}
\text{sign}\left(\frac{\partial I(x)}{\partial \sigma^2(x)}\right) &= \text{sign}\left(\frac{\partial \log I(x)}{\partial \sigma^2(x)}\right) \\
&= \text{sign}\left(-1 - \frac{1}{Z(x)} \frac{z_{\min}}{(\beta(x) - 1)^2} C_1 \frac{1}{\alpha^2}\right) \\
&= -1
\end{aligned}$$

2.

$$\begin{aligned}
&\text{sign}\left(\frac{\partial I(x)}{\partial \sigma_\nu}\right) \\
&= \text{sign}\left(\frac{\partial I(x)}{\partial \sigma^2(x)} \frac{\partial \sigma^2(x)}{\partial \sigma_\nu}\right) \\
&= \text{sign}\left(\frac{\partial I(x)}{\partial \sigma^2(x)}\right) \text{sign}\left(\frac{\partial \sigma^2(x)}{\partial \sigma_\nu}\right) \\
&= -1.
\end{aligned}$$

3.

$$\begin{aligned}
\text{sign}\left(\frac{\partial I(x)}{\partial \alpha}\right) &= \text{sign}\left(\frac{\partial \log I(x)}{\partial \alpha}\right) \\
&= \text{sign}\left(1 + \frac{2}{Z(x)} \frac{z_{\min}}{(\beta(x) - 1)^2} C_1 \frac{\sigma^2(x)}{\alpha^3}\right) \\
&= 1
\end{aligned}$$

4.

$$\begin{aligned}
\text{sign}\left(\frac{\partial I(x)}{\partial \gamma}\right) &= \text{sign}\left(\frac{\partial \log I(x)}{\partial \gamma}\right) \\
&= \text{sign}\left(-1 - \frac{1}{Z(x)} \frac{z_{\min}}{(\beta(x) - 1)^2} \left(\frac{\sigma^2(x)}{\alpha^2} \left(\frac{C_1}{\gamma} + 2\gamma g(\bar{x})\right) - 1\right)\right) \\
&\leq \text{sign}\left(-1 - \frac{1}{Z(x)} \frac{z_{\min}}{(\beta(x) - 1)^2} \left(\frac{\sigma^2(x)}{\alpha^2} \frac{C_1}{\gamma} - 1\right)\right) \\
&= -1
\end{aligned}$$

5.

$$\begin{aligned}
\text{sign}\left(\frac{\partial I(x)}{\partial f_{xx}}\right) &= \text{sign}\left(\frac{\partial \log I(x)}{\partial f_{xx}}\right) \\
&= \text{sign}\left(-\frac{1}{Z(x)} \frac{z_{\min}}{(\beta(x) - 1)^2} \frac{\sigma^2(x)}{\alpha^2} 2\gamma\right) \\
&= -1
\end{aligned}$$

■

Proof. Proposition 4 Investment and Alpha Bijection. For each level of expertise, we have

$$\text{sign}\left(\frac{\partial I(x)}{\alpha}\right) = 1, \text{ for } x \geq \underline{x}.$$

And when α is higher, more experts enter. Thus

$$\frac{\partial I}{\partial \alpha} > 0.$$

■

Proof. Proposition 5. Comparative Statics: Aggregate Demand in Partial Equilibrium and Alpha in General Equilibrium. We use direct calculations. We use 1 to denote a positive sign.

1.

$$\begin{aligned} & \text{sign} \left(\frac{\partial I(x)}{\partial \sigma_\nu} \right) \\ &= \text{sign} \left(\frac{\partial I(x)}{\partial \sigma^2(x)} \frac{\partial \sigma^2(x)}{\partial \sigma_\nu} \right) \\ &= \text{sign} \left(\frac{\partial I(x)}{\partial \sigma^2(x)} \right) \text{sign} \left(\frac{\partial \sigma^2(x)}{\partial \sigma_\nu} \right). \end{aligned}$$

We also have

$$\text{sign} \left(\frac{\partial I(x)}{\partial \sigma^2(x)} \right) = -1$$

Thus for each level of expertise, when total risk is higher, the demand for the complex risky asset is smaller. And, from Equation (18), since $\sigma(x)$ is increasing in σ_ν , when σ_ν is higher, participation is lower.

$$\frac{\partial I}{\partial \sigma_\nu} < 0.$$

2. For each level of expertise:

$$\text{sign} \left(\frac{\partial I(x)}{\partial \gamma} \right) = -1,$$

and from Equation (18), participation is lower. Thus,

$$\frac{\partial I}{\partial \gamma} < 0.$$

3. For each level of expertise:

$$\text{sign} \left(\frac{\partial I(x)}{\partial f_{xx}} \right) = -1,$$

and again from Equation (18), participation is lower. Thus, we also have that

$$\frac{\partial I}{\partial f_{xx}} < 0.$$

■

Proof. Proposition 6. Comparative Statics for Changes in Total Volatility: Individual Sharpe Ratios. Writing out the partial derivative, and using the fact that, for example, $\frac{\partial \log \alpha}{\partial \log \sigma_\nu} =$

$\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu}$, we have that:

$$\begin{aligned} \frac{\partial SR(x)}{\partial \sigma_\nu} &= \frac{\frac{\partial \alpha}{\partial \sigma_\nu} \sigma(x) - \alpha \frac{\partial \sigma(x)}{\partial \sigma_\nu}}{\sigma^2(x)} \\ &= \frac{\alpha}{\sigma(x) \sigma_\nu} \left[\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \right]. \end{aligned}$$

Thus,

$$\frac{\partial SR(x)}{\partial \sigma_\nu} > 0 \text{ iff } \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} > \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}$$

This result illustrates the key role of the elasticity of excess returns vs. the elasticity of effective volatilities with respect to changes in total volatility.

If $\frac{\partial \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}}{\partial x}$ is a constant, we must have either $\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} > \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}$ for all x or $\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} < \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}$ for all x .

If $\frac{\partial \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}}{\partial x} < 0$, and assume there is a cutoff x^* such that

$$\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} = \frac{\partial \sigma(x^*) / \sigma(x^*)}{\partial \sigma_\nu / \sigma_\nu},$$

then for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} < 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} > 0$.

If $\frac{\partial \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}}{\partial x} > 0$, and assume there is a cutoff x^* such that

$$\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} = \frac{\partial \sigma(x^*) / \sigma(x^*)}{\partial \sigma_\nu / \sigma_\nu},$$

then for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} > 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} < 0$. ■

A.2.1 Comparative Statics for Changes in Total Volatility: Participation

Intermediate results and proofs: Bounds on Changes in α . We begin by describing results for bounds on the elasticity of α with respect to changes in total volatility, σ_ν . As we saw in Proposition 6 which describes changes in individual Sharpe ratios as total volatility changes, the relative elasticity of excess returns, $\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu}$, vs. the elasticity of effective volatilities $\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}$ as total volatility changes,

$$\left[\frac{\partial \log \alpha}{\partial \log \sigma_\nu} - \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} \right] \equiv \left[\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \right]$$

plays an important role in our model economy. Both portfolio and participation policies depend on excess returns relative to effective risk. Sharpe ratios at both the individual and market level measure the amount of compensation per unit of risk. Thus, how each of these policies and equilibrium outcomes change as total volatilities change is driven in large part by the relative elasticity of excess returns vs. the elasticity of effective volatilities. If α is highly elastic with respect to changes in total volatility, then many agents will choose to participate even when total volatility is high. On the other hand, if low expertise agents' effective volatilities are very elastic with respect to total volatility, and in particular are more elastic than α is, then they will drop out of the complex asset market when total volatility increases.

Although we cannot solve for α in closed form, we describe bounds on its elasticity with respect to changes in total volatility. These bounds depend on the shape of the elasticity of effective volatility

with respect to total volatility. Intuitively, cases in which participation increases are cases in which the difference in the increase in effective volatility for low and high expertise agents is similar. This is due to the requirement of market clearing. As volatility increases, demand decreases. If all agents' volatilities increase by a similar amount, then for demand to meet supply α must increase enough to satisfy even lower expertise agents. On the other hand, if the change in effective volatility is much smaller for high expertise agents (Case 3 of Proposition 6), then it may be that these agents demand a lot of the risky asset despite the higher volatility. If high expertise agents' demand is high enough, the market can clear at a level of α that does not adequately compensate lower expertise agents.

First, we establish intuition by showing that the percentage change in α has to be large enough to at least satisfy the investors whose risk-return tradeoff deteriorates the least as total volatility increases.

Lemma A.2 *Bounds on Changes in α due to Changes in Total Volatility: All Cases for Proposition 6.* In the equilibrium, we have

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} > l_{\text{inf}}^{\sigma_\nu},$$

where $l_{\text{inf}}^{\sigma_\nu}$ is the lowest elasticity of all participating investors' effective volatility with respect to total volatility

$$l_{\text{inf}}^{\sigma_\nu} \equiv \inf \left\{ \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} \mid x \geq \underline{x} \right\}.$$

Proof. Lemma A.2 Proof by contradiction. Suppose σ_ν is increased by 1%, but the equilibrium α is increased by less than $l_{\text{inf}}^{\sigma_\nu}\%$, that is

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} \leq l_{\text{inf}}^{\sigma_\nu}$$

The condition implies that $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} < 0 \quad \forall x \geq \underline{x}$. We have:

1. Less participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$, \underline{x} increases.

2. Less investment in the complex risky asset:

$$\begin{aligned} & \frac{\partial \log I(x)}{\partial \sigma_\nu} \\ &= \frac{\partial\alpha/\alpha}{\partial\sigma_\nu} - 2 \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} + \frac{1}{Z(x)} \frac{\partial Z(x)}{\partial\sigma_\nu} \\ &= \frac{\partial\alpha/\alpha}{\partial\sigma_\nu} - 2 \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} - \frac{1}{Z(x)} \frac{z_{\min}}{(\beta(x)-1)^2} \frac{\partial\beta(x)}{\partial\sigma_\nu} \\ &= \frac{\partial\alpha/\alpha}{\partial\sigma_\nu} - 2 \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} - \frac{1}{Z(x)} \frac{z_{\min}}{(\beta(x)-1)^2} 2(\beta(x)+\gamma) \left[\frac{\partial\alpha/\alpha}{\partial\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} \right] \\ &= -\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} + \frac{1}{\sigma_\nu} \left[1 + \frac{z_{\min}}{Z(x)} \frac{2(\beta(x)+\gamma)}{(\beta(x)-1)^2} \right] \left[\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} \right] \\ &= \frac{1}{\sigma_\nu} \left\{ -\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} + \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x)+\gamma)}{\beta(x)-1} \right] \left[\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} \right] \right\} \\ &< 0, \text{ for all } x \geq \underline{x}, \end{aligned}$$

where for $\frac{\partial \beta(x)}{\partial \sigma_\nu}$, we use Equation (23). We also use the fact that $Z(x) = z_{\min} \left(\frac{\beta(x)}{\beta(x)-1} \right)$. The fact that we have $\beta(x) \geq 1$ is shown in the proof to Proposition 3.

Define

$$\mathcal{B}(x) = \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1},$$

which describes the elasticity of the mean wealth at expertise level x with respect to total volatility. We can rewrite the partial derivative of expert's investment with respect to total volatility as

$$\frac{\partial \log I(x)}{\partial \sigma_\nu} = \frac{1}{\sigma_\nu} \left\{ -\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} + [1 + \mathcal{B}(x)] \left[\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} \right] \right\} \quad (\text{A.3})$$

Therefore, in the new equilibrium, the total demand for the risky asset is less than the total supply. Contradiction. It must be that

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} > l_{\text{inf}}^{\sigma_\nu}.$$

The last term in the partial derivative for expertise level investment, Equation (A.3), clearly illustrates the important role of the relative elasticity of excess returns vs. the elasticity of effective volatilities. It shows why our bound for the elasticity of α is that its change will be larger than the change in the effective volatility of the agents with the lowest elasticity of effective volatility with respect to total volatility (displayed by the highest expertise agents). ■

The following lemma describes more detailed bounds on the percentage change in α for a given percentage change in total volatility for the case of decreasing elasticities of effective volatility with respect to total volatility (Case 3 of Proposition 6). Case 3 of Proposition 6 is the only case which yields a decline in participation as total volatility increases. The condition for decreasing participation will be closely related to the bounds in Lemma A.3. In particular, we show below that participation increases if Condition 1 of Lemma A.3 holds, but decreases if Condition 2 holds. Intuitively, participation will increase if the change in α is large enough to satisfy lower expertise investors in Case 3, but will decrease otherwise. Lemma A.3 provides bounds on the percentage change in α for a given percentage change in total volatility for Case 3, depending on the elasticity of the lowest expertise agent who participates relative to the elasticity of α , and, depending a condition on the ratio of the partial equilibrium change in demand with respect to total volatility to the partial equilibrium change in demand with respect to α . Intuitively, if, in partial equilibrium, demand is more sensitive to changes in effective volatility than to changes in α , then in general equilibrium participation will decline. We provide a sufficient condition for participation to decline as total volatility increases in Proposition 7 in the main text.

Lemma A.3 *Bounds on Changes in α due to Changes in Total Volatility: Cases 3 in Proposition 6.* In general equilibrium, in Case 3 of Proposition 6, in which $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} \leq 0$, we have that:

1.

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} > l_{\text{sup}}^{\sigma_\nu} \text{ if } l_{\text{sup}}^{\sigma_\nu} < \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)},$$

and

2.

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < l_{\text{sup}}^{\sigma_\nu} \text{ if } l_{\text{sup}}^{\sigma_\nu} > \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)},$$

where

$$\mathcal{B}(x) = \frac{2}{\beta(x)} \frac{(\beta(x) + \gamma)}{\beta(x) - 1}.$$

Proof. Lemma A.3 In Case 3 of Proposition 6, we have $\frac{\partial \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x} < 0$.

Using Equation (A.3) for the change in expertise-level investment with total volatility and accounting for the change in participation we then have for aggregate investment it must be that:

$$\begin{aligned} 0 &= \frac{\partial I}{\partial\sigma_\nu} = \int_{\underline{x}}^{\bar{x}} \frac{\partial I(x)}{\partial\sigma_\nu} d\Lambda(x) - I(\underline{x}) \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial\sigma_\nu} \\ &= \int_{\underline{x}}^{\bar{x}} \left\{ \frac{I(x)}{\sigma_\nu} (1 + \mathcal{B}(x)) \left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} \right) \right\} d\Lambda(x) \\ &\quad - \int_{\underline{x}}^{\bar{x}} \frac{I(x)}{\sigma_\nu} \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda(x) - I(\underline{x}) \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial\sigma_\nu}. \end{aligned} \quad (\text{A.4})$$

Rearranging terms, we have an expression for the change in the participation threshold, weighted by the mass of agents at the threshold, that must be satisfied in order for the risky asset market to clear:

$$\begin{aligned} \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial\sigma_\nu} &= \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\bar{x}} \left\{ \frac{I(x)}{\sigma_\nu} [1 + \mathcal{B}(x)] \left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} \right) \right\} d\Lambda(x) \\ &\quad - \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\bar{x}} \frac{I(x)}{\sigma_\nu} \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda(x). \end{aligned}$$

Collecting terms with the elasticities with respect to α and effective volatility $\sigma(x)$, we can write this as:

$$\begin{aligned} \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial\sigma_\nu} &= \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\bar{x}} \frac{I(x)}{\sigma_\nu} [1 + \mathcal{B}(x)] \left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} \right) d\Lambda(x) \\ &\quad - \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\bar{x}} \frac{I(x)}{\sigma_\nu} [2 + \mathcal{B}(x)] \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda(x). \end{aligned} \quad (\text{A.5})$$

Therefore, from Equation (A.5) we have:

$$\begin{aligned} \frac{\partial \underline{x}}{\partial\sigma_\nu} &> 0 \Leftrightarrow \\ l_{\text{sup}}^{\sigma_\nu} = \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} \Big|_{x=\underline{x}} &> \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} > \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)}. \end{aligned} \quad (\text{A.6})$$

The first inequality comes from the participation condition. We know that $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < l_{\text{sup}}^{\sigma_\nu}$ when participation decreases and \underline{x} increases. The second inequality is from solving Equation (A.5) for the elasticity of α with respect to total volatility in the case that the left hand side of that equation is less

than zero. Using similar arguments, we also have:

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} < 0 \Leftrightarrow l_{\text{sup}}^{\sigma_\nu} = \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} \Big|_{x=\underline{x}} < \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} < \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)}. \quad (\text{A.7})$$

■

To gain intuition for the conditions in Equations (A.6) and (A.7), note that the right hand side is the ratio of the partial equilibrium change in demand with respect to total volatility (holding α and \underline{x} constant), and the partial equilibrium change in demand with respect to α . Consider first the numerator of this ratio. The numerator shows that when σ_ν increases, investors allocate a lower fraction of wealth to the risky asset. We have that $\theta = \frac{\alpha}{\gamma \sigma^2(x)}$. Thus, if all elasticities of effective volatility with respect to total volatility were one, then if σ_ν increased by 1%, investors would allocate 2% less to the risky asset. This leads to the 2 in the numerator. The second term, involving $\beta(x)$ arises because when σ_ν increases, $\beta(x)$ increases, so there is lower total wealth in partial equilibrium. The term

$$\mathcal{B}(x) = \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1}$$

gives the elasticity of the mean wealth at expertise level x with respect to total volatility. The denominator similarly shows how changes in α affects aggregate demand in partial equilibrium, holding \underline{x} and effective volatilities constant. If α increases by 1%, portfolio allocations to the risky asset increase by 1%, and we have the 1 term. Then, we again see the effect of the change in α on mean wealth levels. Now, consider why this ratio matters for the elasticity of α . In Equation (A.6), participation declines because the effect of the change in aggregate demand from increasing effective volatilities is greater than the effect on demand from increasing α . The reverse is true in Equation (A.7).

We can bound the elasticity of α with respect to total volatility more tightly in Case 3 of Proposition 6 as follows:

Lemma A.4 Tighter Bounds on Changes in α due to Changes in Total Volatility: Cases 3 in Proposition 6. When $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} \leq 0$, in the equilibrium, we have,

1.

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} > l_{\text{sup}}^{\sigma_\nu} \text{ if } l_{\text{sup}}^{\sigma_\nu} < \left(1 + \frac{1}{1 + \mathcal{B}(\bar{x})}\right) l_{\text{inf}}^{\sigma_\nu}.$$

and

2.

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} < l_{\text{sup}}^{\sigma_\nu} \text{ if } l_{\text{sup}}^{\sigma_\nu} > \left(1 + \frac{1}{1 + \mathcal{B}(\underline{x})}\right) E \left[\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} \Big| x \geq \underline{x} \right],$$

where

$$\mathcal{B}(x) = \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1}.$$

We show that the percentage change in α for a given percentage change in total volatility will be greater than the highest elasticity of effective volatility with respect to total volatility (displayed by

the participating investor with the lowest expertise) if that highest elasticity is less than a constant times the lowest elasticity, displayed by agents with the highest level of expertise. In other words, if the highest elasticity, displayed by the marginal investor, is close enough to the elasticity of the highest expertise agents, it must be that the change in α is enough to compensate that investor. If not, the change in α will not be great enough to satisfy market clearing.

We also show the converse: The percentage change in α for a given percentage change in total volatility will be less than the highest elasticity of effective volatility with respect to total volatility (displayed by the participating investor with the lowest expertise) if that highest elasticity is greater than a constant near one times the average elasticity over participating investors. In other words, if the marginal investor is different enough from the average, the market can clear despite the fact that their individual Sharpe ratio declines, since the other investors face smaller declines, or increasing individual Sharpe ratios.

Note that the constant will be near one if β is close to one, which it will be as it is the tail parameter from a Pareto distribution. As $\beta(x) - 1$ goes to zero, the denominator of the constant, $\mathcal{B}(\underline{x})$, goes to infinity, so that the entire constant becomes $(1+0)$. Note also that we derive a sufficient condition which is based on the wealth distribution of the highest expertise agents, as using the entire distribution, a mixture of Pareto distributions, is more complicated but would yield similar intuition.

Proof. Lemma A.4 We first derive a lower bound on the ratio of the partial equilibrium change in demand with respect to total volatility (holding α and \underline{x} constant), and the partial equilibrium change in demand with respect to α . This lower bound is the condition bounding $l_{\text{sup}}^{\sigma_\nu}$ in the first statement of Lemma A.4.

$$\begin{aligned}
& \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)} \geq \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)} l_{\text{inf}}^{\sigma_\nu} \\
& = \left(1 + \frac{\int_{\underline{x}}^{\bar{x}} I(x) d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)} \right) l_{\text{inf}}^{\sigma_\nu} \\
& \geq \left(1 + \frac{\int_{\underline{x}}^{\bar{x}} I(x) d\Lambda(x)}{(1 + \mathcal{B}(\bar{x})) \int_{\underline{x}}^{\bar{x}} I(x) d\Lambda(x)} \right) l_{\text{inf}}^{\sigma_\nu} \\
& = \left(1 + \frac{1}{1 + \mathcal{B}(\bar{x})} \right) l_{\text{inf}}^{\sigma_\nu}.
\end{aligned}$$

The first inequality follows because $l_{\text{inf}}^{\sigma_\nu}$ is the lowest elasticity, and it replaces the weighted average elasticity. The next equality groups terms, and the following inequality follows because $\mathcal{B}(x)$ is replaced by $\mathcal{B}(\bar{x})$ and we have that β is decreasing in x . The last equality again groups terms. So, if

$$l_{\text{sup}}^{\sigma_\nu} < \left(1 + \frac{1}{1 + \mathcal{B}(\bar{x})} \right) l_{\text{inf}}^{\sigma_\nu},$$

we have

$$l_{\text{sup}}^{\sigma_\nu} < \left(1 + \frac{1}{\mathcal{B}(\bar{x})} \right) l_{\text{inf}}^{\sigma_\nu} \leq \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)},$$

and this ratio was the bound on the elasticity of α from Lemma A.3. Thus, we have established our tighter bound on the elasticity of α in the first case of Lemma A.4 since we have from Condition 1 in Lemma A.3,

$$l_{\text{sup}}^{\sigma_\nu} = \left. \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} \right|_{x=\underline{x}} < \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu}.$$

Now, we bound the ratio of the partial equilibrium change in demand with respect to total volatility (holding α and \underline{x} constant), and the partial equilibrium change in demand with respect to α from above. This upper bound is the condition bounding $l_{\text{sup}}^{\sigma_\nu}$ in the second statement of Lemma A.4. Consider:

$$\begin{aligned} & \int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x) \\ & \leq \int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) d\Lambda(x) \frac{\int_{\underline{x}}^{\bar{x}} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)}{1 - \Lambda(\underline{x})} \\ & = \int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) d\Lambda(x) E \left[\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \middle| x \geq \underline{x} \right]. \end{aligned}$$

The first inequality follows because the integral $\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)$ puts more weight on the average elasticity of wealth times investment relative to total volatility when $\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}$ is small and less weight when $\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}$ is large, whereas the term after the inequality sign uses an equal weighted average of all effective volatilities. The last equality groups terms. We have now bounded above the numerator in the ratio of the partial equilibrium change in demand with respect to total volatility (holding α and \underline{x} constant), and the partial equilibrium change in demand with respect to α . If we divide through by the partial equilibrium change in demand with respect to α , we have, using similar logic as before:

$$\begin{aligned} & \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)} \\ & \leq \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)} E \left[\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \middle| x \geq \underline{x} \right] \\ & = \left(1 + \frac{\int_{\underline{x}}^{\bar{x}} I(x) d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)} \right) E \left[\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \middle| x \geq \underline{x} \right] \\ & \leq \left(1 + \frac{\int_{\underline{x}}^{\bar{x}} I(x) d\Lambda(x)}{(1 + \mathcal{B}(\underline{x})) \int_{\underline{x}}^{\bar{x}} I(x) d\Lambda(x)} \right) E \left[\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \middle| x \geq \underline{x} \right] \\ & = \left(1 + \frac{1}{1 + \mathcal{B}(\underline{x})} \right) E \left[\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \middle| x \geq \underline{x} \right]. \end{aligned}$$

So, if

$$l_{\text{sup}}^{\sigma_\nu} > \left(1 + \frac{1}{1 + \mathcal{B}(\underline{x})} \right) E \left[\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \middle| x \geq \underline{x} \right],$$

we have

$$l_{\text{sup}}^{\sigma_\nu} > \left(1 + \frac{1}{1 + \mathcal{B}(\underline{x})} \right) E \left[\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \middle| x \geq \underline{x} \right] \geq \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)}.$$

Then, from Condition 2 in Lemma A.3 we can get

$$l_{\text{sup}}^{\sigma_\nu} = \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \bigg|_{x=\underline{x}} > \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu}.$$

■

Comparative Statics for Changes in Total Volatility: Participation Increasing. We now show conditions under which participation increases, i.e. under which the cutoff level of expertise for participation \underline{x} declines, as total volatility increases. In particular, we show that participation increases with total volatility in Cases 1 and 2 of Proposition 6, but only under a tight restriction in Case 3. In Case 3, participation only increases if the elasticity of the effective volatility of the lowest expertise investor is not too different from that of the average participating investor. In other words, participation increases if there is very little difference across expertise levels in the effect of changes in total volatility on effective volatility, so that elasticities are nearly constant, as in Case 1, in which participation always increases as total volatility increases. Notice that the condition restricting the differences in elasticities across investors is the same as Condition 1 in Lemma A.4, which bounds the change in α from below. Thus, participation will increase only if the change in α is large enough, which will be the case if all participating investors face similar changes to their effective volatility as total volatility changes. We discuss the more empirically relevant case for Case 3 of Proposition 6, in which participation declines as total volatility and asset complexity increase, in the text. Recall that the condition for declining participation requires that elasticities vary enough across high expertise and low expertise agents, so that market clearing does not require the participation of lower expertise agents.

Proposition A.1 *Participation Increasing.* Define the entry cutoff \underline{x} ,

$$\underline{x} = \sigma^{-1} \left(\frac{\alpha}{\sqrt{2\gamma f_{xx}}} \right),$$

where $\sigma^{-1}(\cdot)$ is the inverse function of $\sigma(x)$. We have that participation increases with total volatility,

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} < 0$$

if the following conditions hold

1. $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} \geq 0$, (Proposition 6 Cases 1 and 2) or
2. $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0$, (Proposition 6 Case 3) and $l_{\text{sup}}^{\sigma_\nu} < \left(1 + \frac{1}{1+\mathcal{B}(\bar{x})}\right) l_{\text{inf}}^{\sigma_\nu}$.

Proposition A.1 shows that participation increases in Cases 1 and 2 as total volatility increases. The reason is that demand for the complex asset by incumbent experts declines, and new wealth must be brought into the market to clear the fixed supply. However, in Case 3, a tight restriction is required for participation to increase. The restriction is that there is a very small difference between the highest and lowest elasticities. The restriction is tight since β is the tail parameter of a pareto distribution, so that $\beta \approx 1 \forall x$. Then, given the definition of $\mathcal{B}(x)$, this implies that $\mathcal{B}(\bar{x}) \rightarrow \infty$. Intuitively, if the increase in total volatility adversely impacts low expertise agents much more than high expertise agents, then high expertise agents can clear the market without the demand from low expertise investors with substantially deteriorated individual Sharpe ratios.

Proof. Proposition A.1 First,

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} < 0 \text{ iff } \frac{\partial \log \frac{\alpha^2}{\sigma^2(x)}}{\partial \log \sigma_\nu} \Big|_{x=\underline{x}} > 0.$$

We have

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(x)}}{\partial \log \sigma_\nu} \Big|_{x=\underline{x}} = 2 \left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} \Big|_{x=\underline{x}} \right).$$

Therefore

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(x)}}{\partial \log \sigma_\nu} \Big|_{x=\underline{x}} > 0 \text{ iff } \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} > \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}.$$

If $\frac{\partial \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}}{\partial x} \geq 0$, from Proposition A.2 we have

$$\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} > l_{\text{inf}}^{\sigma_\nu} = \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}.$$

If $\frac{\partial \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}}{\partial x} < 0$ and $l_{\text{sup}}^{\sigma_\nu} < \left(1 + \frac{1}{1+\mathcal{B}(\bar{x})}\right) l_{\text{inf}}^{\sigma_\nu}$, from Lemma A.3, we know

$$\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} > l_{\text{sup}}^{\sigma_\nu} = \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}.$$

■

Comparative Statics for Changes in Total Volatility: Participation decreasing in Case 3 of Proposition 6. With these intermediate results in hand, the proof for Proposition 7 follows.

Proof. Proposition 7. Participation decreasing in Case 3 of Proposition 6. First,

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} > 0 \text{ iff } \frac{\partial \log \frac{\alpha^2}{\sigma^2(x)}}{\partial \log \sigma_\nu} \Big|_{x=\underline{x}} < 0.$$

We have

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(x)}}{\partial \log \sigma_\nu} \Big|_{x=\underline{x}} = 2 \left(\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}} \right).$$

Therefore

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(x)}}{\partial \log \sigma_\nu} \Big|_{x=\underline{x}} < 0 \text{ iff } \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} < l_{\text{sup}}^{\sigma_\nu} = \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}.$$

If $\frac{\partial \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu}}{\partial x} < 0$ and $l_{\text{sup}}^{\sigma_\nu} > \left(1 + \frac{1}{1+\mathcal{B}(\underline{x})}\right) E \left[\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big| x \geq \underline{x} \right]$, from Lemma A.3, we know

$$\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} < l_{\text{sup}}^{\sigma_\nu} = \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}.$$

■

We note that the conditions in Proposition A.1 and Proposition 7 are sufficient, but not necessary. As discussed in the main text, we use the tail parameters for the highest and lowest expertise levels since the entire wealth distribution is a mixture of Pareto distributions (a complicated object). We also note that the conditions for increasing vs. decreasing participation in Case 3 are not overlapping, because

$$\left(1 + \frac{1}{1 + \mathcal{B}(\bar{x})}\right) l_{\text{inf}}^{\sigma_\nu} \leq \left(1 + \frac{1}{1 + \mathcal{B}(\underline{x})}\right) E \left[\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big| x \geq \underline{x} \right].$$

A.2.2 Comparative Statics for Changes in Total Volatility: Market Equilibrium Equally Weighted Sharpe Ratio.

The following Proposition describes sufficient conditions for the equally weighted market Sharpe ratio to be increasing with total risk in general equilibrium in all three cases of Proposition 6. It thus extends the results in Proposition 8 in the main text, which covers Case 3. Condition 1 (Equation (A.8)) in Proposition provides an equivalent statement for Condition 2 in Proposition 8, non-representativeness of marginal participants. This equivalent condition substitutes out all equilibrium objects from the Sharpe ratio. However, because we cannot solve for alpha in closed form, the marginal participant \underline{x} still appears. Lemma A.5 proves that these two conditions (Equation (A.8) and Condition 2 in Proposition 8) are equivalent. Following Lemma A.5 and the associated proof is the proof for Proposition A.2. The proof of Proposition 8 (which only applies to Case 3 Proposition 8) is contained at the end of the proof for Proposition A.2, which covers all cases.

Proposition A.2 Comparative Statics for Changes in Total Volatility: Market Equilibrium Equally Weighted Sharpe Ratio Increasing, All Cases. *The equally weighted market Sharpe ratio is increasing with total risk in general equilibrium, i.e.,*

$$\frac{\partial SR^{ew}}{\partial \sigma_\nu} > 0,$$

if:

1. $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} \leq 0$, (Cases 1 and 3 of Proposition 6) and

$$\frac{\frac{\partial \underline{x}}{\partial \sigma_\nu}}{\underbrace{\frac{\lambda(\underline{x})}{1 - \Lambda(\underline{x})}}_{\text{fraction of threshold experts}}} \underbrace{\left(1 - \frac{1}{E \left[\frac{\sigma(\underline{x})}{\sigma(x)} \mid x \geq \underline{x} \right]} \right)}_{\text{difference btw marginal vol and avg}} > \underbrace{- \frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x} \Big|_{x=\underline{x}}}_{\text{sensitivity of threshold vol w.r.t. expertise}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \quad (\text{A.8})$$

or

2. $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} > 0$, (Case 2 of Proposition 6) and

$$\frac{\frac{\partial \underline{x}}{\partial \sigma_\nu}}{\underbrace{\frac{\lambda(\underline{x})}{1 - \Lambda(\underline{x})}}_{\text{fraction of threshold experts}}} \underbrace{\left(1 - \frac{1}{E \left[\frac{\sigma(\underline{x})}{\sigma(x)} \mid x \geq \underline{x} \right]} \right)}_{\text{difference btw marginal vol and avg}} \gg \underbrace{- \frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x} \Big|_{x=\underline{x}}}_{\text{sensitivity of threshold vol w.r.t. expertise}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \quad (\text{A.9})$$

We first provide intuition for the equivalent statement for Condition 2 in Proposition 8, non-representativeness of marginal participants given in Equation (A.8). We then show this equivalence formally in Lemma A.5. Finally, we provide the proof for Proposition A.2.

To gain intuition for the equivalent statement for Condition 2 in Proposition 8, non-representativeness of marginal participants given in Equation (A.8), consider the case in which expertise and complexity are complementary, Case 3 of Proposition 6. Assume that Condition 2 of Proposition 7 is also satisfied so that $\frac{\partial \underline{x}}{\partial \sigma_\nu} > 0$ and we can cancel out this term from both sides of Equation (A.8). The right hand side of Equation (A.8) describes how much the effective volatility of the marginal agent changes with expertise. This term will always be negative before applying the negative sign in the equation; effective volatilities decline with expertise. Thus, the condition in Equation (A.8) states that if the effective volatility of the marginal agent changes substantially as \underline{x} changes, then the right

hand side will be larger, and it is positive. To intuitively see why the right hand side of Equation (A.8) is equivalent to that of Equation (28) in Condition 2 in Proposition 8, consider that we know from the participation condition in Equation (18) that the Sharpe ratio of the marginal agent is always equal to the same constant, determined by parameters. When the curvature of effective volatility is higher, effective volatility declines more moving from the initial participation margin \underline{x} to the new one as total volatility increases. Then, to keep the Sharpe ratio constant at the new participation margin as participation declines, the increase in α is smaller than it would be if effective volatility declined less as the marginal participant changed. Thus, the decline in the Sharpe ratio of the initial marginal participant as total volatility increases is larger for economies in which effective volatility displays more curvature, and we have the equivalence of the right hand sides of Equations (A.8) and (28). To complete the intuition for the condition in Equation (A.8), consider the left hand side of the equation. The term involving the conditional pdf of expertise gives the fraction of experts which reside at the threshold for entry. The magnitude of the next term on the left hand side is determined by the difference between marginal agents' effective volatilities, and the average effective volatility, and it is also always positive. This term is close to its maximum value of one when the second term in the parentheses is near zero, which occurs when the marginal agent's effective volatility is very different from the average. This condition states that, for the SR^{ew} to increase, if the effective volatility of the marginal agent increases substantially, then, either there must be few agents at the participation threshold, or the average Sharpe ratio of participants must be different enough from the Sharpe ratio of the marginal participant. The key to Condition 2 is this middle term. This term, measuring the representativeness of marginal participants' effective volatilities, ensures that the impact of the declining Sharpe ratio of low expertise marginal participants as total volatility increases does not determine, and is not representative of, the overall market Sharpe ratio.

The proof for Proposition prop:ewsrapp appears after the following Lemma A.5, which shows formally that Condition 1 of Proposition prop:ewsrapp (Equation (A.8)) and Condition 2 of Proposition 8 are equivalent, and its associated proof.

Lemma A.5 *Equivalence of Conditions for Non-Representativeness of Marginal Participants* *The sufficient condition for the equally weighted market equilibrium Sharpe ratio to increase with total volatility in Proposition A.2, namely Equation (A.8), is equivalent to Condition 2 in Proposition 8. That is,*

$$\begin{aligned} & \frac{\partial \underline{x}}{\partial \sigma_\nu} \frac{\lambda(\underline{x})}{1 - \Lambda(\underline{x})} \left\{ 1 - \frac{1}{E \left[\frac{\sigma(\underline{x})}{\sigma(x)} \middle| x \geq \underline{x} \right]} \right\} > - \frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x} \Big|_{x=\underline{x}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ \Leftrightarrow & - \frac{\partial [1 - \Lambda(\underline{x})] / [1 - \Lambda(\underline{x})]}{\partial \log \sigma_\nu} \left(1 - \frac{SR(\underline{x})}{SR^{ew}} \right) > - \frac{\partial SR(x) / SR(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}. \end{aligned}$$

Proof. Lemma A.5 We can rearrange some terms in Equation (A.8) using the following equations:

$$\begin{aligned} \frac{\partial [1 - \Lambda(\underline{x})] / [1 - \Lambda(\underline{x})]}{\partial \sigma_\nu / \sigma_\nu} &= - \frac{\lambda(\underline{x})}{1 - \Lambda(\underline{x})} \frac{\partial \underline{x}}{\partial \sigma_\nu / \sigma_\nu}, \\ E \left[\frac{\sigma(\underline{x})}{\sigma(x)} \middle| x \geq \underline{x} \right] &= \frac{E \left[\frac{\alpha}{\sigma(x)} \middle| x \geq \underline{x} \right]}{\frac{\alpha}{\sigma(\underline{x})}} = \frac{SR^{ew}}{SR(\underline{x})}. \end{aligned}$$

Further, using this condition, $\frac{\alpha^2}{\sigma^2(\underline{x})} = 2\gamma f_{xx}$, we can get another equation relating the change in \underline{x} to the change in α as follows:

$$F(\sigma_\nu, \underline{x}) = 2 \log \alpha - 2 \log \sigma(\underline{x}) - \log(2\gamma f_{xx}) = 0.$$

Using the implicit function theorem, we obtain another expression for the change in the participation threshold as total volatility changes:

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} = - \frac{\partial F(\sigma_\nu, \underline{x}(\sigma_\nu)) / \partial \sigma_\nu}{\partial F(\sigma_\nu, \underline{x}) / \partial \underline{x}} = \frac{1}{\sigma_\nu} \frac{\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}}{\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x} \Big|_{x=\underline{x}}}, \quad (\text{A.10})$$

From Equation (A.10), we have that

$$\begin{aligned} \frac{\partial SR(x) / SR(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}} &= \frac{\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}}{\frac{\partial \underline{x}}{\partial \sigma_\nu / \sigma_\nu} \frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x} \Big|_{x=\underline{x}}}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial \underline{x}}{\partial \sigma_\nu} \frac{\lambda(\underline{x})}{1 - \Lambda(\underline{x})} \left\{ 1 - \frac{1}{E \left[\frac{\sigma(\underline{x})}{\sigma(x)} \Big| x \geq \underline{x} \right]} \right\} &> - \frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x} \Big|_{x=\underline{x}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ \Leftrightarrow \underbrace{- \frac{\partial [1 - \Lambda(\underline{x})] / [1 - \Lambda(\underline{x})]}{\partial \sigma_\nu / \sigma_\nu}}_{\text{elasticity of participation}} \underbrace{\left(1 - \frac{SR(\underline{x})}{SR^{ew}} \right)}_{\text{difference between marginal agent and average}} &> \underbrace{- \frac{\partial SR(x) / SR(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}}_{\text{elasticity of marginal SR}}. \end{aligned}$$

■

We are now ready to provide a proof for Proposition 8, showing that under Case 3 of Proposition ??, and a condition for non-representativeness of the marginal participant, the equally-weighted Sharpe ratio increases with total volatility. The proof for Case 3 is included in the proof of Proposition A.2, which covers all cases of Proposition 6.

Proof. Propositions 8 and A.2 Comparative Statics for Changes in Total Volatility: Equally-Weighted Equilibrium Sharpe Ratio Increasing.

For convenience, we consider the partial derivative of the positive transformation $\log SR^{ew}$, namely, the log of the integral over all participants' Sharpe ratios divided by the measure of participants:

$$\begin{aligned} \log SR^{ew} &= \log \frac{\int_{\underline{x}}^{\bar{x}} \frac{\alpha}{\sigma(x)} d\Lambda(x)}{1 - \Lambda(\underline{x})} \\ &= \log \int_{\underline{x}}^{\bar{x}} \frac{\alpha}{\sigma(x)} d\Lambda(x) - \log [1 - \Lambda(\underline{x})] \end{aligned}$$

As σ_ν changes, the equilibrium equally weighted market-level Sharpe ratio will change from several effects. First, each individual effective volatility will increase, according to the elasticity of effective volatility with respect to total volatility at each expertise level. To clear the market, the equilibrium α will increase. Finally, participation will change. Taking these effects together, the change in the equally weighted market-level Sharpe ratio will be the change in the individual Sharpe ratios of each expertise level of investors, weighted by their mass in the distribution of expertise, plus the effect on

participation. By direct calculation, we have:

$$\begin{aligned} & \frac{\partial \log SR^{ew}}{\partial \sigma_\nu} \\ = & \frac{\frac{\alpha}{\sigma_\nu} \int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \right) d\Lambda(x) - \frac{\alpha}{\sigma(\underline{x})} \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu}}{\int_{\underline{x}}^{\bar{x}} \frac{\alpha}{\sigma(x)} d\Lambda(x)} + \frac{1}{1 - \Lambda(\underline{x})} \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu}, \end{aligned}$$

where the first term gives the weighted average change in individual Sharpe ratios, and the second term gives the effect on participation. Note the appearance of the expertise-specific differences in the elasticity of alpha and effective volatility with respect to total volatility. Individual Sharpe ratios increase for individuals for which this difference in elasticities is positive. The larger these differences are on average, then, the more likely it is that the market Sharpe ratio will increase. The sign of $\frac{\partial \underline{x}}{\partial \sigma_\nu}$ depends on which case of Proposition 6 applies. Collecting terms, and cancelling α in the numerator and denominator of the first term, we have:

$$\begin{aligned} & \frac{\partial \log SR^{ew}}{\partial \sigma_\nu} \\ = & \frac{1}{\sigma_\nu} \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \right) d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} + \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} \left\{ \frac{1}{1 - \Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\}. \end{aligned} \quad (\text{A.11})$$

Equation (A.11) contains two changes in equilibrium outcomes, α and \underline{x} . The other variables depend only on parameters, or distribution or functional form assumptions. The market clearing condition can be used to eliminate one of the equilibrium outcomes. In particular, we know that aggregate investment must be unchanged and equated to the aggregate supply of the risky asset. Using Equation (A.3) for the change in expertise-level investment with total volatility and accounting for the change in participation we then have for aggregate investment it must be that:

$$\begin{aligned} 0 &= \frac{\partial I}{\partial \sigma_\nu} = \int_{\underline{x}}^{\bar{x}} \frac{\partial I(x)}{\partial \sigma_\nu} d\Lambda(x) - I(\underline{x}) \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ &= \int_{\underline{x}}^{\bar{x}} \left\{ \frac{I(x)}{\sigma_\nu} (1 + \mathcal{B}(x)) \left(\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \right) \right\} d\Lambda(x) \\ &\quad - \int_{\underline{x}}^{\bar{x}} \frac{I(x)}{\sigma_\nu} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x) - I(\underline{x}) \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu}. \end{aligned} \quad (\text{A.12})$$

Rearranging terms, we have an expression for the change in the participation threshold, weighted by the mass of agents at the threshold, that must be satisfied in order for the risky asset market to clear:

$$\begin{aligned} \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} &= \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\bar{x}} \left\{ \frac{I(x)}{\sigma_\nu} (1 + \mathcal{B}(x)) \left(\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \right) \right\} d\Lambda(x) \\ &\quad - \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\bar{x}} \frac{I(x)}{\sigma_\nu} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x). \end{aligned} \quad (\text{A.13})$$

Collecting terms with the elasticities with respect to α and effective volatility $\sigma(x)$, we can write this as:

$$\begin{aligned} & \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ = & \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\bar{x}} \frac{I(x)}{\sigma_\nu} (1 + \mathcal{B}(x)) \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x) - \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\bar{x}} \frac{I(x)}{\sigma_\nu} (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x) \end{aligned} \quad (\text{A.14})$$

Equation (A.14) essentially states that the change in demand for the complex asset from the change in participation must be met by an offsetting change in demand by the remainder of participants in the

complex risky asset market. There are two parts to the change in demand of participants above the threshold. First, there is an increase in demand of 1% for every 1% increase in α , multiplied by the elasticity of wealth with respect to changes in volatility. Second, there is a decrease in demand of 2% for every 1% increase in effective volatility, again weighted by the elasticity of wealth with respect to changes in volatility. Grouping terms, and then plugging this expression for $\lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu}$ into $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu}$, we obtain

$$\begin{aligned}
& \frac{\partial \log SR^{ew}}{\partial \sigma_\nu} \\
&= \frac{1}{\sigma_\nu} \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \right) d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} + \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} \left\{ \frac{1}{1 - \Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \\
&= \frac{1}{\sigma_\nu} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right) + \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} \left\{ \frac{1}{1 - \Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \\
&= \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} \left\{ \frac{1}{\sigma_\nu} + \left\{ \frac{1}{1 - \Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{\sigma_\nu I(\underline{x})} \int_{\underline{x}}^{\bar{x}} I(x) (1 + \mathcal{B}(x)) d\Lambda(x) \right\} \\
&\quad - \left\{ \frac{1}{1 - \Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{\sigma_\nu I(\underline{x})} \int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x) \\
&\quad - \frac{1}{\sigma_\nu} \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)}.
\end{aligned}$$

Therefore, $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} > 0$ if and only if

$$\begin{aligned}
& \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} \\
&> \frac{\left\{ \frac{1}{1 - \Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x) + \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)}}{\left\{ \frac{1}{1 - \Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x) + 1}.
\end{aligned}$$

We divide through by $\frac{1}{1 - \Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)}$ and rearrange to get an expression that will allow us to more easily bound the elasticity of α . We then have $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} > 0$ if and only if

$$\begin{aligned}
& \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} \\
&> \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x) + \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \frac{1}{\left\{ \frac{1}{1 - \Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} I(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x) + \frac{1}{\left\{ \frac{1}{1 - \Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} I(\underline{x})}}. \quad (\text{A.15})
\end{aligned}$$

Now, in order to derive an exact expression for α as a function of parameters, consider again the term $\frac{\partial \underline{x}}{\partial \sigma_\nu}$. The participation condition in Equation (18) can be used to show that the individual Sharpe ratio of the marginal agent is a constant function of the coefficient of relative risk aversion and

the participation maintenance cost. Then, using this condition, $\frac{\alpha^2}{\sigma^2(\underline{x})} = 2\gamma f_{xx}$, we can get another equation relating the change in \underline{x} to the change in α as follows:

$$F(\sigma_\nu, \underline{x}) = 2 \log \alpha - 2 \log \sigma(\underline{x}) - \log(2\gamma f_{xx}) = 0.$$

Using the implicit function theorem, we obtain another expression for the change in the participation threshold as total volatility changes:

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} = - \frac{\partial F(\sigma_\nu, \underline{x}(\sigma_\nu)) / \partial \sigma_\nu}{\partial F(\sigma_\nu, \underline{x}) / \partial \underline{x}} = \frac{1}{\sigma_\nu} \frac{\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}}{\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x} \Big|_{x=\underline{x}}}, \quad (\text{A.16})$$

where we see the difference in the elasticity of alpha and effective volatility with respect to total volatility for the marginal agent in the numerator. Since effective volatility is decreasing in expertise, the effective volatility for the marginal agent decreases as \underline{x} increases, and the denominator is negative. Thus, participation declines and \underline{x} increases in total volatility the smaller the elasticity of α is relative to the elasticity of the effective volatility of the marginal agent. Then, from Equations (A.14) and (A.16), we have another expression for the elasticity of α with respect to changes in total volatility:

$$\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} = \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x) - \frac{I(x)\lambda(x)}{\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x}} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}}{\int_{\underline{x}}^{\bar{x}} I(x) (1 + \mathcal{B}(x)) d\Lambda(x) - \frac{I(x)\lambda(x)}{\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x}} \Big|_{x=\underline{x}}}. \quad (\text{A.17})$$

Now, from Equation (A.15) we have a lower bound on what the elasticity of α must be in order for $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} > 0$. From Equation (A.17) we have an expression for α . Thus, we have that $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} > 0$ if and only if the following condition on the lower bound on the elasticity of α holds:

$$\begin{aligned} & \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x) - \frac{I(x)\lambda(x)}{\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x}} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}}{\int_{\underline{x}}^{\bar{x}} I(x) (1 + \mathcal{B}(x)) d\Lambda(x) - \frac{I(x)\lambda(x)}{\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x}} \Big|_{x=\underline{x}}} \\ & > \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x) + \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \frac{1}{\left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\}} \frac{1}{I(\underline{x})}}}{\int_{\underline{x}}^{\bar{x}} I(x) (1 + \mathcal{B}(x)) d\Lambda(x) + \frac{1}{\left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\}} \frac{1}{I(\underline{x})}}}. \quad (\text{A.18}) \end{aligned}$$

We will show that this condition matches the conditions provided in Propositions A.2 and 8. To continue the proof, we define a simplified notation for the condition in Equation (A.18), namely we write that $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} > 0$ if and only if:

$$\frac{a + c}{b + d} > \frac{a + e}{b + f}, \quad (\text{A.19})$$

where we define:

$$\begin{aligned}
a &= \int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x), \\
b &= \int_{\underline{x}}^{\bar{x}} I(x) (1 + \mathcal{B}(x)) d\Lambda(x), \\
c &= - \frac{I(x) \lambda(x) \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}}}{\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x}}, \\
d &= - \frac{I(x) \lambda(x) \Big|_{x=\underline{x}}}{\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x}}, \\
e &= \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \frac{1}{\left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{I(\underline{x})}}, \\
f &= \frac{1}{\left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{I(\underline{x})}}.
\end{aligned}$$

Comparing the right and left hand sides of the condition under which the equally weighted market Sharpe ratio is increasing,

$$\frac{a+c}{b+d} > \frac{a+e}{b+f},$$

note that the difference is that the left side essentially includes $\frac{c}{d}$, while the right hand side includes $\frac{e}{f}$. Both sides include $\frac{a}{b}$. Recall from the proof of Lemma A.3, and Equations (A.6) and (A.7), that $\frac{a}{b}$ is the ratio of the partial equilibrium change in demand with respect to total volatility (holding α and \underline{x} constant), and the partial equilibrium change in demand with respect to α . To gain further intuition for our condition, we can use the expressions for $\frac{c}{d}$ and $\frac{e}{f}$, and simplify to show that:

$$\frac{c}{d} = \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} \Big|_{x=\underline{x}} \quad \text{and} \quad \frac{e}{f} = \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_\nu / \sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)}.$$

Thus, in loose terms what is needed for the equally weighted market Sharpe ratio to increase is that the elasticity of the marginal agent is sufficiently larger than the weighted average over all elasticities, where the weights are increasing in expertise. This condition essentially ensures that α increases by enough to outweigh the increase in effective volatilities of agents with lower expertise levels who still choose to participate. We begin by describing sufficient conditions for $\frac{a+c}{b+d} > \frac{a+e}{b+f}$, in the three cases of Proposition 6, and depending on whether participation is increasing or decreasing in total volatility. Then, using these results, we derive a sufficient condition covering all cases.

Case 1 of Proposition 6, and Case 3 of Proposition 6 with Increasing Participation:

First, we have

$$\frac{e}{f} \leq \frac{c}{d} < \frac{a}{b}.$$

To see this, note that in Case 1 of Proposition 6, elasticities are constant in x and so $\frac{e}{f} = \frac{c}{d}$. The last inequality follows from the fact that when participation is increasing, Equation (A.7) from the proof of Lemma A.3 shows that $\frac{a}{b}$ is larger than the elasticity of the marginal agent, and hence larger than the elasticity of all agents. In Case 3, we have declining elasticities, generating the first inequality. The last inequality again follows from Equation (A.7), which holds when participation increases.

We can show that $\frac{a+c}{b+d} > \frac{a+e}{b+f}$ if

$$d < f.$$

This is true because

$$\begin{aligned} \frac{a+c}{b+d} > \frac{a+e}{b+f} &\Leftrightarrow (a+c)(b+f) > (a+e)(b+d) \\ &\Leftrightarrow af + cb > ad + be \\ &\Leftrightarrow a(f-d) > b(e-c) \\ &\Leftrightarrow f > d \text{ and } \frac{a}{b} > \frac{e-c}{f-d}. \end{aligned}$$

We relate the condition that $d < f$ to underlying parameters and collect the results for all Cases of Proposition 6 at the end of the proof.

Case 2 of Proposition 6: First, we prove that

$$\frac{a}{b} = \frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)} > \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)} = \frac{e}{f}.$$

To show this, consider the function:

$$\begin{aligned} H(K(x)) &= \frac{\int_{\underline{x}}^{\bar{x}} K(x) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} K(x) d\Lambda(x)} \\ &= \int_{\underline{x}}^{\bar{x}} \frac{K(x) d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} K(x) d\Lambda(x)} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} dx. \end{aligned}$$

This function, $H(K(x))$, represents a “weighted” average of elasticities, with weights given by $K(x)$. Suppose we have

$$K_1(x) = K_2(x) K_3(x),$$

where we assume that K_3 is monotonically increasing in x . Because we have that $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} \geq 0$, and K_3 is monotonically increasing in x , we have that the weights $K_1(x)$ underweight lower elasticities and overweight higher elasticities relative to the weights $K_2(x)$. Therefore,

$$H(K_1(x)) \geq H(K_2(x)).$$

Note that we can write:

$$\underbrace{I(x) (1 + \mathcal{B}(x))}_{K_1(x)} = \frac{\overbrace{1}^{K_2(x)}}{\sigma(x)} \underbrace{[I(x) \sigma(x) (1 + \mathcal{B}(x))]}_{K_3(x)},$$

and further we have that:

$$I(x) \sigma(x) (1 + \mathcal{B}(x)) = \frac{\alpha}{\gamma \sigma(x)} Z(x) (1 + \mathcal{B}(x)),$$

which is monotonically increasing, as assumed for $K_3(x)$. Thus, we have:

$$\frac{\int_{\underline{x}}^{\bar{x}} I(x) (2 + \mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)} > \frac{\int_{\underline{x}}^{\bar{x}} I(x) (1 + \mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} [I(x) (1 + \mathcal{B}(x))] d\Lambda(x)} > \frac{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x)}, \quad (\text{A.20})$$

where the first inequality is because we replace a one with a two in the numerator, and the second inequality is because we showed that $H(K_1) > H(K_2)$.

So, we have $\frac{c}{d} < \frac{e}{f} < \frac{a}{b}$. The first inequality follows from the fact that in Case 2 of Proposition 6, because elasticities increase with expertise, the lowest expertise agent has the smallest elasticity of effective volatility with respect to total volatility. Thus, $\frac{c}{d} = l_{\text{inf}}^{\sigma\nu}$. The second inequality follows from Equation (A.20).

Using these results, we can derive the following inequalities

$$\frac{a+e}{b+f} < \frac{a}{b}, \text{ and } \frac{a+c}{b+d} < \frac{a}{b},$$

Therefore, we have $\frac{a+c}{b+d} > \frac{a+e}{b+f}$ if

$$d \ll f.$$

This is true mathematically because if d is small enough relative to f , then the value of $\frac{a+c}{b+d}$ will be closer to $\frac{a}{b}$ than $\frac{a+e}{b+f}$ will be.

Case 3 of Proposition 6 with Decreasing Participation. In Case 3 of Proposition 6 we have that $\frac{c}{d} > \frac{e}{f}$. Because in Case 3 elasticities decline with expertise, the lowest expertise agent has the largest elasticity with respect to total volatility. Thus, $\frac{c}{d} = l_{\text{sup}}^{\sigma\nu}$, which will be larger than the weighted average $\frac{e}{f}$. We also have, then, that $\frac{c}{d} > \frac{a}{b}$ from Equation (A.6). Given these relationships, we proceed to show that a sufficient condition for

$$\frac{a+c}{b+d} > \frac{a+e}{b+f}$$

is that

$$d > f.$$

We consider each case for $\frac{a}{b}$ vs. $\frac{e}{f}$ separately, since we do not know their relative values in Case 3:

1. First, if $\frac{e}{f} \leq \frac{a}{b} < \frac{c}{d}$, then

$$\frac{a+e}{b+f} \leq \frac{a}{b}, \text{ and } \frac{a+c}{b+d} > \frac{a}{b},$$

thus

$$\frac{a+c}{b+d} > \frac{a+e}{b+f},$$

without any restriction on d relative to f .

2. Second, if $\frac{c}{d} > \frac{e}{f} > \frac{a}{b}$ and $d > f$, we have

$$\frac{a}{b} < \frac{e}{f} < \frac{c-e}{d-f},$$

where the second inequality is from the following algebraic argument:

$$\frac{e}{f} < \frac{c-e}{d-f} \Leftrightarrow ed < cf \Leftrightarrow \frac{c}{d} > \frac{e}{f}.$$

Then, we have, using these results and our assumption that $d > f$:

$$\begin{aligned} & d > f \text{ and } \frac{a}{b} < \frac{c-e}{d-f} \\ \Rightarrow & a(f-d) > b(e-c) \\ \Leftrightarrow & af + cb > ad + be \\ \Rightarrow & af + cb + cf > ad + be + ed \\ \Leftrightarrow & (a+c)(b+f) > (a+e)(b+d) \\ \Leftrightarrow & \frac{a+c}{b+d} > \frac{a+e}{b+f}. \end{aligned}$$

Combining Conditions for Cases 1-3 of Proposition 6 In Case 1 and Case 3 with increasing participation, we need

$$d < f.$$

In Case 3 with decreasing participation, we need

$$d > f.$$

Combined with participation effects, we can use the following one condition:

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} d > f \frac{\partial \underline{x}}{\partial \sigma_\nu}.$$

That is,

$$\begin{aligned} & -\frac{\partial \underline{x}}{\partial \sigma_\nu} \frac{I(x) \lambda(x)}{\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x}} \Big|_{x=\underline{x}} > \frac{1}{\left\{ \frac{1}{1-\Lambda(\underline{x})} - \int_{\underline{x}}^{\bar{x}} \frac{1}{\sigma(x)} d\Lambda(x) \right\} I(\underline{x})} \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ \Leftrightarrow & -\frac{\partial \underline{x}}{\partial \sigma_\nu} \frac{\lambda(x)}{\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x}} \Big|_{x=\underline{x}} > \frac{1}{\frac{1}{1-\Lambda(\underline{x})} \left\{ 1 - \frac{1}{E\left[\frac{1}{\sigma(x)} | x \geq \underline{x}\right]} \right\}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ \Leftrightarrow & \frac{\partial \underline{x}}{\partial \sigma_\nu} \frac{\lambda(\underline{x})}{1 - \Lambda(\underline{x})} \left\{ 1 - \frac{1}{E\left[\frac{\sigma(\underline{x})}{\sigma(x)} | x \geq \underline{x}\right]} \right\} > -\frac{1}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x} \Big|_{x=\underline{x}} \frac{\partial \underline{x}}{\partial \sigma_\nu}. \end{aligned}$$

Thus, the condition that $d > f$ is equivalent to the alternative for Condition 2 in Proposition 8.

■

Numerical Algorithm Because α and I form a bijection (Proposition 4 provides conditions for which they are one to one and onto), for any given supply of the complex risky asset, we can solve for the market equilibrium α in the following steps:

1. Choose an upper and a lower bound for α , namely α_1 and α_2 , ($\alpha_1 > \alpha_2$).
2. Let $\alpha = \frac{\alpha_1 + \alpha_2}{2}$, and compute the total demand for the risky asset

$$\int \lambda(x) I(x) dx$$

3. If $S - \int \lambda(x) I(x) dx < -10^{-4}$, let $\alpha_1 = \alpha$ and back to step 1; if $S - \int \lambda(x) I(x) dx > 10^{-4}$, let $\alpha_2 = \alpha$ and back to step 1; otherwise, STOP.

Comparative Statics on z_{min} When z_{min} is higher, the mean wealth will be higher for each level of expertise. As a result, the market α and Sharpe Ratio will be smaller, and participation is lower. In our baseline calibration, we set $z_{min} = 0.01$. We can see from Table A.0, around the small value for z_{min} we use in our calibration, our results are fairly insensitive to this choice.

Table A.0: Comparative Statics on z_{min}

	α	Sharpe ratio	Participation
$z_{min} = 0.005$	0.0311	0.5820	0.4997
$z_{min} = 0.01$	0.0300	0.5671	0.4893
$z_{min} = 0.02$	0.0271	0.5265	0.4614

B Appendix: Additional Results

Additional Results: Fund flows related to expertise. For illustrative purposes, we present the optimization problem and wealth dynamics for a specification in which wealth grows exogenously as an increasing, concave, and bounded function of expertise, x , to show that our main results are preserved under this specification. In this case, the experts' optimization problem is given by:

$$V^x(w(t), x) = \max_{c^x(x,t), T, \theta(x,t)} \mathbb{E} \left[\int_t^T e^{-\rho(s-t)} u(c^x(x, s)) ds + e^{-\rho(T-t)} V^n(w(t), x) \right] \quad (\text{B.1})$$

$$\text{s.t. } dw(t) = [w(t)(r_f + \theta(x, t)\alpha(t) + i(x)) - c^x(x, t) - F_{xx}] dt + w(t)\theta(x, t)\sigma(x)dB(t), \quad (\text{B.2})$$

where $i(x)$ represents the exogenous investment flow into the experts' wealth. We have that where $y^x(x)$ is given by:

$$y^x(x) = \left[\frac{(\gamma - 1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma - 1)i(x)}{\gamma} + \frac{(\gamma - 1)\alpha^2}{2\gamma^2\sigma^2(x)} \right]^{-\gamma} \quad \text{and} \quad (\text{B.3})$$

$$y^n(x) = \left[\frac{(\gamma - 1)r_f + \rho}{\gamma} \right]^{-\gamma}. \quad (\text{B.4})$$

Furthermore, the wealth of experts evolves according to the law of motion:

$$\frac{dw(t)}{w(t)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{i(x)}{\gamma} + \frac{(\gamma + 1)\alpha^2(t)}{2\gamma^2\sigma^2(x)} \right) dt + \frac{\alpha(t)}{\gamma\sigma(x)} dB(t) \quad (\text{B.5})$$

Finally, investors choose to be experts if the excess return earned per unit of wealth exceeds the maintenance cost per unit of wealth:

$$\frac{i(x)}{\gamma} + \frac{\alpha^2(t)}{2\sigma^2(x)\gamma} \geq f_{xx}. \quad (\text{B.6})$$

Suppose $i(x)$ is increasing, concave, and bounded above, we can obtain an equivalent $\tilde{\sigma}(x)$, where

$$\frac{1}{\tilde{\sigma}^2(x)} = \frac{2i(x)}{\alpha^2} + \frac{1}{\sigma^2(x)},$$

and our remaining analysis follows the same lines as in our main specification.

Additional Results: Efficiency of joint wealth expertise distribution We study the effect of the efficiency of the joint distribution of wealth and expertise on equilibrium pricing. In particular, we demonstrate that the equilibrium required excess return on the complex risky asset is decreasing in the amount of wealth commanded by agents with higher levels of expertise. The proof appears in the Appendix.

Proposition B.1 *If $\frac{\partial\sigma(x)}{\partial x} < 0$, and Λ_1 exhibits first-order stochastic dominance over Λ_2 , $I(\Lambda_1) \geq I(\Lambda_2)$. As a result $\alpha(\Lambda_1) < \alpha(\Lambda_2)$.*

The wealth distribution at each expertise level is a Pareto distribution with an expertise-specific tail parameter. By shifting the distribution of expertise rightward, leading to a new distribution with a relatively larger fraction of higher expertise investors, relatively more wealth will reside with agents

with higher expertise. Thus, with any rightward shift, the joint distribution of wealth and expertise becomes more efficient. Moreover, because the wealth distribution at higher expertise levels exhibits fatter right tails, there is an additional direct effect on overall wealth from a rightward shift in the distribution of expertise. Accordingly, Proposition B.1 shows that if the density of experts shifts to the right, then demand for the complex risky asset will increase, and the required equilibrium excess return will decrease. The equilibrium excess return is decreasing in the amount of wealth which resides in the hands of agents with higher expertise. Note that this result can also be interpreted to state that in asset markets in which higher levels of expertise are more widespread, or less rare, equilibrium required returns will be lower. We argue that the scarcity of relevant expertise is increasing with asset complexity, again implying a higher α in more complex markets.

Proof. Proposition B.1 We have

$$\text{sign} \left(\frac{\partial I(x)}{\partial x} \right) = \text{sign} \left(\frac{\partial I(x)}{\partial \sigma(x)} \frac{\partial \sigma(x)}{\partial x} \right) = 1$$

And, using integration by parts, we obtain:

$$\begin{aligned} I(\Lambda_1) - I(\Lambda_2) &= \int [\lambda_1(x) - \lambda_2(x)] I(x) dx \\ &= -I(\underline{x}) [\Lambda_1(\underline{x}) - \Lambda_2(\underline{x})] - \int \frac{\partial I(x)}{\partial x} [\Lambda_1(x) - \Lambda_2(x)] dx \\ &> 0 \end{aligned}$$

■

Additional Results: Extension to Two Complex Assets In this section, we describe how the intuition from comparative statics across stationary equilibria can be applied to a single economy with multiple complex assets. In particular, we consider an economy with two risky assets with different total volatilities given by σ_ν^L and σ_ν^H , where $\sigma_\nu^H > \sigma_\nu^L$. We assume that agents can only invest in one risky asset; they must specialize to take advantage of their expertise. The following Proposition gives a condition for which higher expertise agents specialize in the more complex risky asset, with total volatility σ_ν^H .¹⁴

Proposition B.2 *In a two complex asset economy, higher expertise agents choose to specialize and invest in the more complex asset, with total volatility σ_ν^H if and only if*

$$\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0.$$

Proof. Proposition B.2 Suppose we have two risky assets with different total volatility, σ_ν^H and σ_ν^L . Because each agent can only invest in one risky asset, we only need to compare the value function of investing in σ_ν^H versus the value function of investing in σ_ν^L . Suppressing the notation ν , denote the effective volatilities for each asset as $\sigma^H(x)$ and $\sigma^L(x)$.

From the Proposition 1, in a stationary equilibrium, we have

¹⁴The analysis can be extended to multiple risky assets, and we omit the conditions under which agents with higher expertise will invest in σ_ν^L .

$$\begin{aligned}
V^n(w, x) &= y^n(x) \frac{w^{1-\gamma}}{1-\gamma}, \\
V_H^x(w, x) &= y_H^x(x) \frac{w^{1-\gamma}}{1-\gamma}, \\
V_L^x(w, x) &= y_L^x(x) \frac{w^{1-\gamma}}{1-\gamma},
\end{aligned}$$

where $y^n(x)$, $y_H^x(x)$ and $y_L^x(x)$ are given by:

$$\begin{aligned}
y_H^x(x) &= \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha_H^2}{2\gamma^2\sigma_H^2(x)} \right]^{-\gamma} \\
y_L^x(x) &= \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha_L^2}{2\gamma^2\sigma_L^2(x)} \right]^{-\gamma} \\
y^n(x) &= \left[\frac{(\gamma-1)r_f + \rho}{\gamma} \right]^{-\gamma}.
\end{aligned}$$

Consider an equilibrium in which all higher expertise investors, $x \geq \hat{x}$, invests in σ_ν^H , medium expertise, $\underline{x} \leq x < \hat{x}$, invests in σ_ν^L , and low expertise, $\underline{x} > x$, invests in the risk free asset only. Then, the following conditions must be satisfied,

$$\begin{aligned}
\frac{\alpha_H^2}{\sigma_H^2(x)} &\geq \frac{\alpha_L^2}{\sigma_L^2(x)}, \text{ for all } x \geq \hat{x}, \\
\frac{\alpha_H^2}{\sigma_H^2(x)} &\leq \frac{\alpha_L^2}{\sigma_L^2(x)}, \text{ for all } \underline{x} \leq x < \hat{x}, \\
\frac{(\gamma-1)\alpha_L^2}{2\gamma^2\sigma^2(\underline{x})} &= f_{xx}.
\end{aligned}$$

Since \hat{x} is indifferent between investing in σ_ν^H and σ_ν^L , the equilibrium α 's must satisfy

$$\alpha_H = \frac{\sigma_H(\hat{x})}{\sigma_L(\hat{x})} \alpha_L = \left(1 + \frac{\sigma_H(\hat{x}) - \sigma_L(\hat{x})}{\sigma_L(\hat{x})} \right) \alpha_L.$$

It is straightforward to further show that:

$$\begin{aligned}
\frac{\alpha_H^2}{\sigma_H^2(x)} &\geq \frac{\alpha_L^2}{\sigma_L^2(x)}, \text{ for all } x \geq \hat{x} \Leftrightarrow \frac{\frac{\sigma_H(x) - \sigma_L(x)}{\sigma_L(x)}}{\sigma_H - \sigma_L} \leq \frac{\frac{\alpha_H - \alpha_L}{\alpha_L}}{\sigma_H - \sigma_L}, \\
\frac{\alpha_H^2}{\sigma_H^2(x)} &\leq \frac{\alpha_L^2}{\sigma_L^2(x)}, \text{ for all } \underline{x} \leq x < \hat{x} \Leftrightarrow \frac{\frac{\sigma_H(x) - \sigma_L(x)}{\sigma_L(x)}}{\sigma_H - \sigma_L} \geq \frac{\frac{\alpha_H - \alpha_L}{\alpha_L}}{\sigma_H - \sigma_L},
\end{aligned}$$

which is true if the elasticity of effective volatility is decreasing, i.e. if

$$\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0.$$

The model can be solved in an analogous way to each single asset economy. There are now four equations and four unknowns. The participation conditions are:

$$\begin{aligned}
\frac{\alpha_H^2}{\sigma_H^2(\hat{x})} &= \frac{\alpha_L^2}{\sigma_L^2(\hat{x})}, \text{ and} \\
\frac{(\gamma-1)\alpha_L^2}{2\gamma^2\sigma^2(\underline{x})} &= f_{xx}.
\end{aligned}$$

The equilibrium $\alpha_L, \alpha_H, \hat{x}$, and \underline{x} , must satisfy these two conditions, as well as the two market clearing

conditions equating total demand to total detrended supply of each asset. ■

Alternative Models: We present a nested model with participation costs, risk aversions, and effective volatility that all depend on expertise. In particular, we assume that the maintenance cost of participation depends on expertise level, $f(x)$, and investor risk aversions are given by $\gamma(x)$. Everything else remains the same as in our baseline model setup. We use this nested model to compare the results from a shock to total volatility on alpha, Sharpe ratios, and participation.

Proposition B.3 Value and Policy Functions for Alternative Models: *The value functions are given by*

$$V^x(w(t), x) = y^x(x, t) \frac{w(t)^{1-\gamma(x)}}{1-\gamma(x)} \quad (\text{B.7})$$

where $y^x(x)$ and $y^n(x)$ are given by:

$$y^x(x) = \left[\frac{(\gamma(x)-1)(r_f - f(x)) + \rho}{\gamma(x)} + \frac{(\gamma(x)-1)\alpha^2}{2\gamma(x)^2\sigma^2} \right]^{-\gamma(x)} \quad \text{and} \quad (\text{B.8})$$

The optimal policy functions $c^x(x, t)$, and $\theta(x)$ are given by:

$$c^x(x, t) = [y^x(x)]^{-\frac{1}{\gamma(x)}} w(t), \quad (\text{B.9})$$

$$\theta(x, t) = \frac{\alpha(t)}{\gamma(x)\sigma^2}. \quad (\text{B.10})$$

Furthermore, the wealth of experts evolves according to the law of motion:

$$\frac{dw(t)}{w(t)} = \left(\frac{r_f - f(x) - \rho}{\gamma(x)} + \frac{(\gamma(x)+1)\alpha^2(t)}{2\gamma(x)^2\sigma^2} \right) dt + \frac{\alpha(t)}{\gamma(x)\sigma} dB(t) \quad (\text{B.11})$$

Finally, investors choose to be experts if the excess return earned per unit of wealth exceeds the maintenance cost per unit of wealth:

$$\frac{\alpha^2(t)}{2\sigma^2\gamma(x)} \geq f(x). \quad (\text{B.12})$$

First, consider a model with a fixed effective volatility and risk aversion which is the same for all agents, but allow agents with higher expertise to have lower maintenance costs of participation. Although this model can generate non-participation by lower expertise agents, conditional on participating all investors will have identical portfolio choices, which implies identical leverage ratios. This is clearly counterfactual, as leverage ratios for MBS funds in the Hedge Fund Research Database (Fund Identifier File) vary from 1 to 2.5. This model would also imply identical returns across funds that participate, which is inconsistent with the data in Table 2, which shows considerable idiosyncratic risk and heterogeneity in fund performance.

Ceteris Paribus, this model also implies an increase in participation upon the realization of an unexpected volatility shock. This can be easily shown by contradiction. If there is a shock to market volatility, alpha must increase to clear the market. Now, suppose effective volatility increases by 10%, but alpha increases by less than 10%. Then, all incumbents will reduce their portfolio choice, $\theta(x)$. From the participation condition in Equation (B.12), there will be fewer experts. But, in this case, the total demand for the risky asset will be lower and the market will fail to clear. Thus, in the new equilibrium, the increase in alpha must exceed the increase in effective volatility in order for the market to clear. Given this, Equation (B.12) implies an increase in participation.

To generate the observed decline in participation, the participation costs would have to change with volatility. If participation costs increase with volatility, but less so for higher expertise agents, one can generate exit. However, this model would still generate (counterfactual) identical leverage ratios, portfolio choice, and performance across investors.

Next, consider the model with coefficients of relative risk aversions that decline with expertise. This model can generate heterogeneity in portfolio choice, and non-participation by agents with lower expertise. However, following the same proof by contradiction, following a shock to volatility there will be an increase in participation. To elaborate, suppose effective volatility increases by 10%, but alpha increases by less than 10%. Then, all incumbents will reduce their portfolio choice, $\theta(x)$. From the participation condition in Equation (B.12), there will be fewer experts. But, in this case, the total demand for the risky asset will be lower and the market will fail to clear. Thus, in the new equilibrium, the increase in alpha must exceed the increase in effective volatility in order for the market to clear. Given this, Equation (B.12) implies an increase in participation.

Comparing these models to our model with heterogeneity in effective volatilities, or idiosyncratic risk, only our model can jointly explain the observed patterns of (1) exit following a shock to volatility (2) heterogeneity in portfolio choice (equivalently, in leverage ratios) and (3) heterogeneity in fund performance.

Finally we also note that there would still be alpha in a long run equilibrium of our model with one level of expertise and idiosyncratic risk. In this case, α net of fees would have to make investors indifferent between being an expert and not:

$$\alpha = \sqrt{2\gamma f_{xx}}\sigma_\nu. \tag{B.13}$$

This benchmark, however, would imply no heterogeneity in portfolio choice (leverage) in equilibrium, and no heterogeneity in performance. This is inconsistent with the data in Figure 1 and Table 1.